

Physics 505: HW4 (due Wed, 22nd October, 2008)

1. FW 3.17: Using Lagrange multipliers to determine when a mass detaches from a sphere.
2. FW 3.18: A slightly more involved problem using Lagrange multipliers, where we resolve the non-holonomic constraints by patching together calculations with holonomic constraints.

When the question asks “Show that the subsequent motion can be reduced to explicit integrals” what I want you to do is write down the integrals needed to explicitly solve for the motion, but *not* to do these integrals. In particular, you don’t need to do them in order to answer the last part of the question (“Does the ladder ever lose contact with the floor?”)

3. In this exercise you are asked to fill in steps in the derivation the result of which was discussed briefly in class. It concerns the result that a velocity dependent potential can be used to describe the Lorentz force on a charged particle.

(a) Show that using the potential

$$V = q[\phi(\vec{r}, t) - \vec{v} \cdot \vec{A}(\vec{r}, t)] \quad (1)$$

in the Euler-Lagrange equations leads to the correct Lorentz force

$$\vec{F} = q(\vec{E} + \vec{v} \times \vec{B}). \quad (2)$$

Here the units are such that

$$\begin{aligned} \vec{E} &= -\nabla\phi - \frac{\partial\vec{A}}{\partial t}, \\ \vec{B} &= \nabla \times \vec{A}. \end{aligned}$$

(b) Recall the form of gauge transformations

$$\vec{A} \longrightarrow \vec{A} + \vec{\nabla}\Lambda; \quad \phi \longrightarrow \phi - \frac{\partial\Lambda}{\partial t},$$

with $\Lambda(\vec{r}, t)$ an arbitrary function. This does not change \vec{E} or \vec{B} , and thus does not affect the dynamics. It does, however, change L .

- i. Show that L changes by a total time derivative of the form dF/dt , where F is a function of $\{x_i\}$ and t . Determine F .
 - ii. We know from Hamilton’s principle that this cannot change the equations of motion. Show this result instead by manipulation of partial derivatives.
4. Deriving Lagrange’s equation (FW eq. 15.17). This applies to constrained motion, where what I call q_β^R , $\beta = 1, n - k$ are the unconstrained coordinates (FW just call these q_α). Lagrange’s equations for such motion are:

$$\frac{d}{dt} \frac{\partial T}{\partial \dot{q}_\beta^R} - \frac{\partial T}{\partial q_\beta^R} = Q_\beta \equiv \sum_{i=1}^n \frac{\partial x_i}{\partial q_\beta^R} F_i^{(a)}, \quad (3)$$

where T is the kinetic energy, and $F_i^{(a)}$ are the Cartesian components of the “applied” force, which does not include the forces of constraint. The point of this equation is that it holds for all types of applied forces, not just for conservative forces. The quantities Q_β are called “generalized forces”.

The aim of this problem is to derive eq. (3), following a different route than that used in FW. This will hopefully help elucidate the physics behind this equation.

- (a) Show, using Newton’s laws and d’Alembert’s principle that

$$\sum_{i=1}^n \frac{\partial x_i}{\partial q_\beta^R} \left(\frac{d}{dt} \frac{\partial T}{\partial \dot{x}_i} - \frac{\partial T}{\partial x_i} \right) = Q_\beta, \quad (4)$$

for all β . *Hint:* Consider what quantity one obtains when multiplying this equation by δq_β^R .

- (b) Adapt the manipulations in the last part of question 5 of HW3 to derive Lagrange’s equation.
5. An application of Lagrange’s equation, eq. (3). Having done the work to derive it, let’s get some payback. One application is to provide an alternative derivation of fictitious forces in accelerating and rotating frames, and we’ll work out the rotating case here.

Let (ρ, θ, z) be cylindrical coordinates in a basis which itself is rotating about the z axis relative to an inertial frame. The angle of this rotation is $\phi(t)$. Let (x, y, z) be Cartesian coordinates in the inertial frame (so that the z ’s are the same in the two bases).

- (a) Determine expressions for x and y in terms of ρ , θ and $\phi(t)$.
- (b) Using this determine the explicit form of the left-hand sides of Lagrange’s equations for $q_\beta^R = \rho$ and θ , assuming a single particle of mass m . (Don’t worry about the z equation, which is unchanged in form from that in the original coordinates.) Check that you find terms corresponding to the various fictitious forces that we discussed previously.
- (c) Interpret the right-hand sides of these two Lagrange equations—i.e. what is their expression in terms of Forces and how would you describe them?