

Physics 505: HW3 (due 15th October, 2008)

1. This problem asks you to complete the calculation of the dynamics of the Foucault pendulum that we started in class. The problem is completely worked out in FW, using a trick specific to the particular case. The aim here is to use a more pedestrian, but more general, perturbative approach.

For small oscillation angles, the pendulum bob's motion is approximately horizontal, and is described by the equations derived in class:

$$\begin{aligned}\ddot{x} &= -\omega_P^2 x + 2\omega \cos \theta \dot{y}, \\ \ddot{y} &= -\omega_P^2 y - 2\omega \cos \theta \dot{x},\end{aligned}$$

where $\omega_P = \sqrt{g/\ell}$ is the angular frequency of the oscillations of the pendulum in the absence of the Coriolis force (with ℓ the pendulum length), $\omega \ll \omega_P$ is the Earth's rotation angular frequency, and θ is the "co-latitude".

- (a) Explain briefly why, in order to understand the motion of the pendulum, it is sufficient to analyze a single "swing", starting at rest on one side (say at $x = -x_0$, $y = 0$, $\dot{x} = \dot{y} = 0$) and ending at rest on (almost) the opposite side.
 - (b) Carry out this single-swing analysis as follows:
 - i. Determine $x(t)$ assuming $\omega = 0$ —i.e. work at zeroth order in ω for the x motion.
 - ii. Insert this result into the equation for $y(t)$ and solve. This ignores the feedback of \dot{y} on the x equation, but this is an effect of $O(\omega^2)$ and thus very small.
 - iii. Determine the ending position of the swing, and from this the rate of change of the pendulum's oscillation plane.
2. We will be using the calculus of variations as part of Lagrangian mechanics. Here is a problem of statics whose solution requires similar methods.

Consider a string of uniform density which is draped over two pulleys and allowed to reach its equilibrium configuration. The pulleys are fixed at heights y_1 and y_2 above the ground, and are separated by a horizontal distance Δx . They rotate without friction, and you can treat their radii as vanishingly small. The string is very long, so that it hangs down on the "outside" of the pulleys onto the ground on both sides, with plenty of excess string lying on the ground. It moves without friction on the ground. The final idealization is that it costs no energy to make the string have curvature—i.e. it is completely flexible.

The aim of the problem is to determine the stationary configuration or configurations of the string. We are interested in possible solutions in which the string does not touch the ground in the "middle", i.e. in the region between the two pulleys. (There is always the less interesting solution in which the string hangs vertically down on both sides of both pulleys, and lies on the ground in between.)

- (a) What is the configuration of the string on the outside of the pulleys? Explain briefly.
- (b) Formulate the problem as an extremization of an appropriate quantity.
- (c) Use the calculus of variations to determine the differential equation satisfied by the string configuration $y(x)$. Be sure to specify the boundary conditions.

- (d) Solve the differential equation to determine the general form of $y(x)$, and give the equations which fix the integration constants. *Hint:* use the first integral discussed in class. There is no need to work out explicit expressions for these constants.

One could continue by asking what are the conditions on y_1 , y_2 and Δx such that there are solutions, and whether these are stable. But that's for another day . . .

3. FW 3.1. An application of Lagrangian methods to a constrained system in which the constraints are accounted for by use of suitable generalized coordinates.
4. FW 3.4. More practice setting up the Lagrangian in a constrained system, and determining the equations of motion. Note that you are not asked to solve these equations. To avoid unnecessary complications, assume that the motion of the bob is two-dimensional (“along the slope”). Also, I suggest using as coordinates the distance d of the center of the plate *up the slope* and the angle θ between the pendulum arm and the supporting rod. Finally, you should imagine that the “rod” is set up in such a way that the pendulum can pass “through” it.

When the question asks you to “discuss the limit $\alpha \rightarrow 0$ ”, you should determine the equations of motion for $\alpha = 0$ by elementary means, and compare to your general result. (It is a fun exercise to solve for the motion in this case, but you don't need to do this for the problem.)

5. Practice with partial derivatives. Show, by careful manipulation of partial derivatives, that if the Euler-Lagrange equation holds for $L(\{x_i\}, \{\dot{x}_i\}, t)$, i.e.

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{x}_i} = \frac{\partial L}{\partial x_i} \quad (1)$$

for all $i = 1, n$, then an equation of the same form holds when L is expressed in terms of any generalized coordinates $y_j = y_j(\{x_i\}, t)$ [note that, by the definition of generalized coordinates, there is no dependence on the $\{\dot{x}_i\}$]. Here both i and j run from 1 to n . In class we showed this using the action and the calculus of variations, but the aim here is to take a pedestrian approach to check the result.

Next consider the case that there are k holonomic constraints such that the motion is determined by a reduced number of independent coordinates $q_\alpha = q_\alpha(\{x_i\}, t)$, with $\alpha = 1, n - k$. Using your previous work, derive an expression for

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{q}_\alpha} = \frac{\partial L}{\partial q_\alpha} \quad (2)$$

in terms of the quantities

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{x}_i} = \frac{\partial L}{\partial x_i}. \quad (3)$$

(This result, applied to T rather than L , will be used in class to discuss the case of non-conservative external forces.)