Outline

March 15, 2007

0.1 Fluids and solids

Static properties of fluids were well understood in the seventeenth century, with work by Torricelli and Pascal built on ancient work by Archimedes and others. A fluid, such as the air around us or water, is characterized by its *pressure* and *density*. Pressure gives a force acting on any body in the fluid, and it is the resultant force that determines if a body will float or sink in the fluid. Density is the average mass per unit volume. The relation between pressure and density determines the *bulk modulus* of the fluid.

Fluid dynamics is much more difficult, even though the basic ideas were known to Newton (hence the term 'Newtonian fluids'). D. Bernoulli, Navier, Stokes, Prandtl, Kármán and many others worked out the theory of fluid flow, which was necessary for the design of airplanes and fast surface vehicles. Few problems in fluid dynamics have exact solutions in closed form, and general principles had to be understood before the analysis of experiments gave much guidance. I will also talk a little about the properties of solids for the sake of contrast. The discussions of fluid dynamics and of solids are necessarily superficial, since they give brief surveys of complicated subjects.

Thermal physics

In the eighteenth century heat was widely supposed to be a conserved fluid, which would flow from warmer regions to cooler regions, and which would be released when matter was ground up, for example by a saw or by a grindstone, and which was also bound up or released by chemical reactions. For some purposes this was good enough, and, for example, the work by Fourier on heat conduction did not depend on the nature of heat. Benjamin Thompson, better known as Count Rumford, showed that heat could be extracted from matter in almost unlimited amounts by working on the material. He advocated the idea that heat was a manifestation of matter in random motion; this was the idea that was developed in statistical mechanics by L. Boltzmann and J. Clark Maxwell.

0.2 Temperature and thermal equilibrium

Bodies in equilibrium are characterized by a temperature, and heat tends to flow from warmer (higher temperature) to cooler (lower temperature) regions, so that the temperature is made more uniform. Thermometers are devices to measure temperature, often by means of thermal expansion of their material, but also by means of voltages generated, or their color, or by using some other temperature-dependent property. In the early days rather different temperature scales could be measured if the thermometers were different, but the behavior of ideal gases provides a preferred temperature scale that has universal validity in a wide temperature range. As the course develops so should your understanding of temperature.

In this section the most important concepts are *temperature*, *equation of state* and *thermal expansion*.

0.3 Heat and work

Measurement of the mechanical equivalent of heat by Joule established the relation between heat and work — 4.2 joules per calorie. As J. Mayer suggested, heat is not separately conserved, but is just converted into other forms of energy. This gives the **first law of thermodynamics**. Some concepts we develop here are *quantity of heat*, *heat capacity*, *specific heat*, and *latent heat*.

In this section we shall also study the transport of heat, by *thermal conduction*, by *convection*, by *radiation*, and by other related means.

0.4 Kinetic theory and statistical mechanics

Kinetic theory is based on the idea that, although there is no chance of using mechanics to predict the detailed motion of a large number of interacting atoms or molecules, their average behavior can be predicted. One powerful result is that in thermal equilibrium the kinetic energy of the center of mass of a molecule depends only on the temperature. This leads to the idea of *equipartition of energy*, which helps us to understand observed specific heats of materials, and to a derivation of the observed dependence of the density of gases on pressure and temperature —*Boyle's Law* and *Charles' Law*.

We will develop some of the simpler aspects of the theory of *probability*, and learn about the *Boltzmann distribution*.

0.5 Second law of thermodynamics

The principle that you cannot get something for nothing leads to remarkably powerful predictions, in particular that *entropy* is related to temperature as pressure is related to density for a substance. From the statistical point of view, entropy is a measure of the disorder of the molecules in a material.

One of the consequences of the second law is that not all the energy associated with heat is available. The available energy is characterized as *free energy*, which is a combination of energy and entropy.

0.6 Phase equilibrium

One of the simplest systems to which thermodynamics can be applied is a mixture of two phases of the same substance, such as ice and water or liquid water and water vapor. More complicated situations arise in mixtures of different substances, such as in chemical equilibrium, or in binary alloys. Some questions we shall look at are why most liquids solidify under pressure, but ice melts. Ordinary solid–liquid mixtures get warmer when pressure is applied, but ice with water gets colder, as does a mixture of liquid and solid ^{3}He at very low temperatures.

Some mathematical tools

We will be dealing with functions of several variables, so will be using partial derivative notation, and doing simple manipulations with partial derivatives.

Integrals along a path in a two or three-dimensional space will be used. Surface or volume integrals may be mentioned occasionally, but will not be essential.

Some elementary probability arguments will be used when we come to discuss statistical mechanics, but the concepts we need will be explained as we go along.

1 Fluids and Solids

1.1 Gases, liquids and solids

Gases or vapors (I do not distinguish between gases and vapors, although some people use the words in a different sense), and liquids are both known as *fluids*, because they have no resistance, over the long term, to changing their shape. A gas will fill any container it is put in, while a liquid will usually fill the lower part of a container, while the rest of the container is filled with air or the vapor of the liquid. On any surface in the fluid there is a force normal to the surface, proportional to the area of the surface, and the normal force per unit area is the *pressure*. We will see later (in sec. 3) that this force is due to the molecules of the fluid bouncing off the solid surface.

In this section of the course I mostly use a continuum description of these states of matter, but later we will relate this in more detail to a molecular description of the materials. In gases the molecules are far apart from one another, and their high kinetic energy keeps them apart, so that gravity is not strong enough to confine them to the bottom of a container. In liquids, the molecules are close together, and are kept together by the attractive forces that act between any pair of neutral molecules, provided that they are not so close to one another that they begin to overlap.

A solid, which maintains its shape, is much more complicated, and there are *stress* forces on it, defined in terms of six different components of the force on it, with different directions on different planes. The distortions that such forces produce in the solid are known as the components of *strain*, and the relations between stress and strain are given by the *moduli of elasticity*.

The molecules of a solid are not only close to one another, like the molecules of a liquid, but they are arranged in a regular pattern, the *crystal lattice*. This regular arrangement of the molecules prevents one layer of atoms from slipping easily over another, and this is what is responsible for the rigidity of a solid.

Liquids are difficult to compress, gases easy. There is a big difference between the compressibility of air and water — about a factor of 10^4 — but dense gases and hot liquids are more similar to one another than are dilute gases and cold liquids. At sufficiently high temperatures, above the *critical temperature*, the distinction between the two phases of the same material disappears. In this course liquids will be taken to be incompressible, since pressures of the order of 10^9 Nm^{-2} are needed to compress a liquid a lot, but the compressibility of gases always has to be taken into account.

1.2 Density

The average density of an object is its mass divided by its volume. For a material that is reasonably homogeneous, the density is a characteristic of the material, dependent on conditions such as temperature, pressure, chemical composition.

- For water at 0 C and atmospheric pressure, density is 999.8 kg/m^3;
- for mercury at 20 C, density is 13546 kg/m^3;
- for liquid helium at low pressure, density is 125 kg/m^3;
- for liquid hydrogen, density is 71 kg/m^3;
- for air at 20 C, atmospheric pressure, density is 1.2 kg/m^3.

Liquids, like solids, are rather incompressible. It takes a pressure of 200 atmospheres to change the density of water by 1%, and 2600 atmospheres to change density of mercury by 1%. Gases increase in density by 1% when the pressure is increased by 1%.

We think of the atoms in liquids as being packed rather close together, while in gases the molecules are far apart.

For inhomogeneous bodies, such as a ship or a human body, we can write the density as a function of position, $\rho(\mathbf{r})$. The average density can then be written as

$$\rho_{\mathrm{ave}} = \frac{\int \rho(\mathbf{r})\, d^3r}{\int d^3r} \ .$$

Our bodies have an average density a little less than that of water. Our bones are considerably denser than water, but fat is less dense, and our lungs are much less dense.

1.3 Pressure

Atmospheric pressure is the force per unit area that holds up the atmosphere. Torricelli invented a barometer that demonstrates and measures this pressure. He showed that 760 mm of mercury or 10 m of water have the same weight per unit area as the entire atmosphere. This is shown by having a long tube sealed at the top end, opening to a bowl at its bottom end which acts as a reservoir for the mercury (or water), as is shown in fig. 1. On the top (sealed) end there is only the pressure of the vapor, which is very small for mercury, and on the reservoir at the bottom there is the

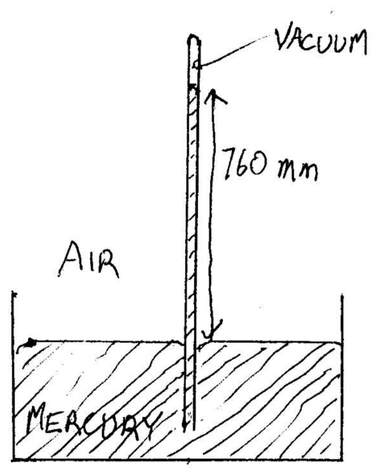

Figure 1: A mercury barometer has atmospheric pressure acting on the reservoir at the bottom. This pressure supports the weight of mercury in the sealed tube, above which there is only the very small pressure of mercury vapor

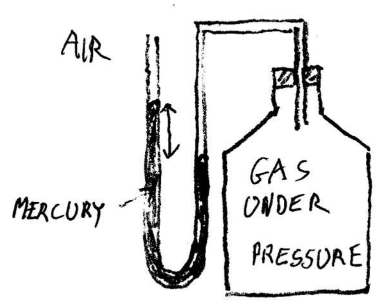

Figure 2: A manometer, open to a gas or fluid container at one end, and to the atmosphere at the other end, measures the gauge pressure of the fluid, its pressure minus atmospheric pressure.

pressure of the atmosphere. At sea level the pressure is 1.0×10^5 Nm^{-2}, which corresponds to a mass of 1.0×10^4 $\mathrm{kg\,m}^{-2}$.

A lot of different units are used for pressure. The SI (Système Internationale) unit is known as *pascal*, abbreviated as Pa, equivalent to Nm^{-2}. The approximate sea-level atmospheric pressure is known as a *bar*. The pressure corresponding to 1 mm of mercury is known as a *Torr*. We have

$$10^5 \text{ Pa} = 10^5 \text{ Nm}^{-2} = 1 \text{ bar} \approx 760 \text{ Torr} \approx 15 \text{ lb/in}^2 \approx 30 \text{ in Hg}\,.$$

Since everything on the earth's surface is under atmospheric pressure, which acts on all surfaces however oriented, pressures are often measured relative to atmospheric pressure, and such pressures are called *gauge pressures.* One meter deep in fresh water the absolute pressure is 1.1×10^5 Pa, and the gauge pressure is 0.1×10^5 Pa. Manometers, such as the one shown in fig. 2, measure gauge pressure, as do tire pressure gauges. If you are told your blood pressure is 120 over 65, is this a gauge pressure or an absolute pressure? What units is it measured in? Does it make a significant difference whether the measuring cuff is at the same height as your heart, or at a different level?

A saying of Aristotelian physics was "Nature abhors a vacuum", but the seventeenth century scientists joked that nature only abhorred the first thirty feet of vacuum.

Pascal took a barometer up a mountain and showed that the pressure got lower, because there was less atmosphere above him. If you do not believe it go up a mountain pass with a thin polycarbonate drink model, drink half the bottle on the pass, seal it up, and come back to sea level.

We have this pressure outside the body, inside our lungs, and inside our chest cavities. How does an unprotected diver go down 30 m and survive? It must be done by having enough air in the chest to start the dive that it can be compressed without the ribs collapsing.

Pressure is force per unit area, so by applying force to a small area, as in a bicycle pump, you can generate a high pressure, say 5 bar gauge pressure, in the tires of a bicycle. What force is needed to produce this pressure using a piston 2 cm in diameter? With a hydraulic jack you can easily lift an SUV. With the help of the leverage of a pump arm you apply a high pressure to oil in a fairly small diameter cylinder. This pressure acting on a much larger cylinder lifts the vehicle. As with all such devices, although the force you apply is quite small, you have to move your arm a long way to do the work involved in raising the SUV.

Note that pumps can push water uphill as high as the mechanical strengths of the pistons, cylinders, and valves allow, but can only pull it up 10 m, because any greater height will lead to negative pressure at the top, and the water will vaporize.

1.4 Boyle's law and the isothermal atmosphere

Boyle's law for an ideal gas is

$$pV = \text{constant} , \tag{1}$$

where p, V are the pressure and volume of a fixed quantity of gas at constant temperature. Robert Boyle found this by doing early experiments with air pumps. This equation implies that density and pressure are proportional to one another, so Boyle's law can be written as

$$\rho(p) = \alpha(T)p , \tag{2}$$

where $\alpha(T)$ is a temperature dependent constant characteristic of the chemical composition of the gas. If you go up a mountain, say 3000 m high, you take in less oxygen in each breath, so you need to breathe more often when you are exercising. One result of this is that you flush out the CO_2 from your body more quickly, which can make you feel unwell (mountain sickness) and can make it difficult to sleep. In a plane the cabin pressure is usually reduced significantly below 1 bar.

The pressure of the atmosphere is what is supporting the weight of the atmosphere. If you consider a column of air of cross-section A between heights

z and $z + \delta z$, its mass is approximately $\rho(z)A\,\delta z$, its weight is $\rho(z)Ag\,\delta z$, acting downwards, and the net pressure force on it

$$[p(z) - p(z + \delta z)]A \approx -\frac{dp}{dz}\delta z A \ ,$$

acting upwards. For the pressure force to balance the gravitational force we must have

$$\frac{dp}{dz} = -\rho g = -\alpha(T)pg \quad \text{or} \quad \frac{dp}{p} = -\alpha(T)g\,dz \ . \tag{3}$$

If we assume the temperature is independent of height, so that α is just a constant equal to ρ_0/p_0, this can be integrated to give

$$\ln p = -\frac{\rho_0}{p_0}gz + \text{constant}, \quad \text{or} \quad p = p_0 e^{-\rho_0 gz/p_0} \ , \tag{4}$$

where p_0 is the pressure at $z = 0$. This gives the way the pressure of an atmosphere, assumed isothermal (temperature independent of height), varies with height. Since the density of air at 1 atm. pressure, 20 C temperature, is 1.2 kg/m^3, $p_0/\rho_0 g$ is approximately 8.6 km, and this is the height at which the pressure would drop by a factor of e. We shall discuss later why the temperature drops as you go up in a real atmosphere, but this isothermal atmosphere gives you a rough idea of what happens in the real atmosphere.

1.5 Archimedes and all that

The story of Archimedes and the crown is a good one to think about. Hieron of Syracuse gave his goldsmith some gold to make a crown with, but suspected the goldsmith of cheating him by keeping some of the gold, and replacing it with an equal weight of silver. Hieron asked Archimedes to check it out. Archimedes knew that an mixture of gold and silver would have a different density from the original gold. It was easy to measure the mass of the crown, but even the world's best geometer did not know how to work out the volume of the crown accurately enough. One day when he was worrying about how to solve this problem he got into the bath, and noticed the water which had been displaced by his body spilling over the edge. He was so excited by this that he jumped out of his bath and ran round shouting "Eureka — I have found it!"

The important principle here is that, since both water and solids are incompressible, **when a solid body**, such as Archimedes or the crown, **is submerged in water, the water will rise by just the amount that corresponds to the volume of the body**. Thus all that Archimedes

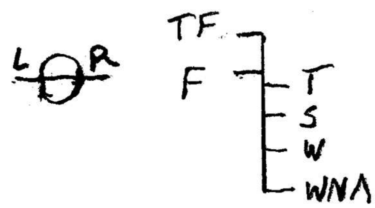

Figure 3: The load line on a ship show the level to which a ship may be safely loaded. The levels, in increasing order of water density, are Tropical Fresh, Fresh, Tropical, Summer, Winter, and Winter North Atlantic.

needed to do was to put the crown in a full container of water, and to measure the volume of water that spilled over the the top of a container. Since a liquid will take up any shape, he could use a container of simple shape, such as a cylinder, in which to do the measurement.

There is a refinement of this story which most of the books favor. Archimedes did not need to measure the water displaced, but he could infer it by first weighing the crown in air, and then by hanging it on a wire an weighing it when it was completely under the water. The water that is displaced exerts a pressure on all surfaces of the crown, and this exerts an upward force which was just the force needed to support the weight of the water displaced. Because of this upward force, **the measured weight of the crown is reduced by the weight of water displaced**. This is a more accurate, but usually slightly slower, way of determining the density.

Remember that density is mass per unit volume, not weight per unit volume.

An object **floats** in a liquid if its average density is less than the density of the liquid. It floats freely, displacing the amount of water that has the same total mass. The weight of the floating body is supported by the pressure force of the surrounding liquid, which is just the same as it was when the fluid took up the space now occupied by the part of the floating body below the level of the liquid. The mass of a laden ship is referred to as its "displacement", because that is how it is measured. The appropriate loading line has to be adjusted according to the temperature and salinity of the water — the line is marked as TF, F, T, S, W, WNA, in order of increasing water density. The denser the water, the lower is the line to which the ship may be loaded. This force exerted by the pressure of the surrounding liquid is known as the *buoyant force*. If the average density of the object is greater than the density of the liquid, the object will sink down until the weight of the object is also supported by the solid below the liquid as well as the insufficient buoyant force. In deep water there is also some change in water density, due to changing temperature, pressure and salinity, but the effects are relatively small. Also the solid object may be compressed by the increase of pressure with depth, so that its buoyancy gets less. This is usually unimportant, but in a "diving bell", which is a bell-shaped container open at the bottom, the air that is trapped in it gets compressed as the bell goes down, so that the buoyancy decreases. In the seventeenth century, going down in a diving bell was a substitute for SCUBA diving.

For balloons and other things floating in the atmosphere we have to take account of the large variation of the density with altitude, both because of changing pressure and changing temperature. A hot air balloon rises until it reaches an altitude at which its average density (including the load it is carrying) is equal to the density of the atmosphere at that height.

1.6 Bulk modulus

The *bulk modulus* K of a fluid is defined as minus the logarithmic derivative of the pressure with respect to volume

$$K = -V \frac{dp}{dV} \,. \tag{5}$$

For an ideal gas satisfying Boyle's Law, differentiation of Eq. (1) gives $(dp/dV)V + p = 0$, so the bulk modulus K is just the pressure p. In a liquid it is much larger, 2×10^9 Pa for water, 26×10^9 Pa for mercury.

1.7 Elastic solids

A solid resists changes of shape as well as changes in density. Not only does it have a bulk modulus defined by Eq. (5), but it also has a *rigidity modulus*. If two opposite faces of a solid cube are acted on by a positive pressure $+F$ (compressive stress), and another pair of opposite faces are acted on by a negative pressure $-F$ (tensile stress), leaving the third pair of faces free, this subjects the solid to what is called a *shearing stress*. Stress, like pressure, is a force per unit area, but the direction of the force and the plane on which it acts must be defined. This produces a distortion of the third pair of faces which is called a *shearing strain*. The ratio of the shearing stress to strain is the rigidity modulus. The stresses in these two cases are shown in fig. 4 (a)

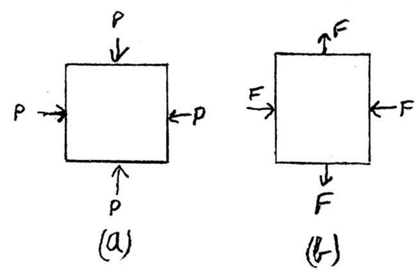

Figure 4: An isotropic compressive stress P is shown in (a) acting on all three pairs of faces of a cube, producing the uniform compressive strain that defines the bulk modulus. A compressive stress on one pair of faces combined with an equal and opposite tensile stress on another pair of faces is shown in (b). This produces a shearing strain that changes the shape but not the volume of an isotropic solid.

and (b). The stress shown in fig. 4(b) has distorted the square cross section to a rectangular shape.

Other elastic constants can be defined in similar ways. For example, *Young's modulus* is the ratio of a tensile force to the relative extension it produces. When a rod is under tension it usually shrinks in the transverse direction while it expands in the longitudinal direction, and the ratio of the lateral shrinking to the longitudinal stretching is known as the *Poisson ratio.* These elastic constants can be calculated from the bulk modulus and the rigidity modulus provided the solid is isotropic. For a single crystal, or for a fibrous material like wood the situation is more complicated.

1.8 Principles of fluid flow

The principles of fluid mechanics are quite simple, but they lead to complicated problems. The simple general principles are:

1. Mass is conserved;

2. Force is equal to rate of change of momentum;

3. There is an equation of state for the fluid — its density depends on pressure, temperature, composition, etc.

4. Forces are those due to pressure, gravity, viscosity, etc.

In one space dimension the flux of mass out of the region between X_1 and X_2 is

$$\rho(X_2)v(X_2) - \rho(X_1)v(X_1) = \int_{X_1}^{X_2} \frac{\partial}{\partial x}[\rho(x,t)v(x,t)]dx \ .$$

This must equal the rate of decrease of mass in the interval, which is

$$-\int_{X_1}^{X_2} \frac{\partial}{\partial t}\rho(x,t)dx \ ,$$

so the equation of mass conservation is

$$\frac{\partial}{\partial x}[\rho(x,t)v(x,t)] + \frac{\partial}{\partial t}\rho(x,t) = 0 \ .$$

This argument can be generalized to three dimensions, where it gives

$$\frac{\partial}{\partial x}[\rho(\mathbf{r},t)v_x(\mathbf{r},t)] + \frac{\partial}{\partial y}[\rho(\mathbf{r},t)v_y(\mathbf{r},t)] + \frac{\partial}{\partial z}[\rho(\mathbf{r},t)v_z(\mathbf{r},t)] + \frac{\partial}{\partial t}\rho(\mathbf{r},t) = 0 \ . \tag{6}$$

The first three terms represent the rate of flow of mass, per unit volume, out of a small volume, while the last term represents the rate of increase of mass per unit volume in a small volume.

The equation of motion looks complicated, because the expression for rate of change of momentum is complicated. The momentum change is produced by pressure and gravitational forces, and by *viscosity*, which is proportional to the rate at which one layer of the fluid is slipping past the neighboring layers. Whereas the shearing stress in a solid is proportional to the shearing strain, the viscous stress in a fluid is proportional to the *rate of strain.*

If we consider an incompressible fluid (constant density ρ) in steady motion, so that the velocity at a given point does not change with time, we can define *streamlines* in terms of the direction of the velocity $\mathbf{v}$; fluid that starts on a given streamline remains on that streamline. In the following discussion I take the fluid to be flowing from left to right. We consider a narrow tube of liquid surrounded by streamlines, and use s to denote the distance along this tube, $A(s)$ to denote the cross-sectional area of the tube, $p(s)$ for the pressure, and $\mathbf{v}(s)$ for the fluid velocity along the tube; then the speed $|\mathbf{v}(s)|$ is equal to ds/dt. For a given area $A(s)$ normal to the streamlines, all the fluid in the tube within a distance $\delta t\, ds/dt$ to the left of it will pass through the area in the time interval δt. This fluid has volume $A(s)\delta t\, ds/dt$ and mass $\rho A(s)\delta t\, ds/dt$. Conservation of matter requires that, in steady flow, the same mass of fluid passes through each cross-section of the tube in each time interval, so this must be a constant, independent of s. For an incompressible fluid, ρ is constant, so $A(s)ds/dt = A(s)|\mathbf{v}(s)|$ is constant, and the speed of flow is inversely proportional to the cross-sectional area.

The work done by pressure in time δt on a slab of fluid in the tube, initially between the areas $A(s)$ and $A(s+\delta s)$, is

$$W = p(s)A(s)v(s)\delta t - p(s+\delta s)A(s+\delta s)v(s+\delta s)\delta t \ ,$$

since $p(s)A(s)$ is the force acting to the right on the left of the slab, and $-p(s+\delta s)A(s+\delta s)$ is the force acting to the right on the right of the slab. Since I have argued that $A(s)v(s)$ is constant along the tube for an incompressible fluid, this can be written as

$$W \approx -\frac{dp}{ds}Av\,\delta s\,\delta t = -\frac{dp}{ds}\frac{ds}{dt}\delta t\,A\,\delta s = -\frac{dp}{dt}\delta t\,A\,\delta s \ .$$

The work goes to change the sum of the kinetic and potential energies in this length δs of the tube, which is

$$\left[\frac{1}{2}\rho v^2 + \rho g z\right] A\delta s \ ,$$

where z is the height of the tube. Since $A\,\delta s$ can be divided through both these equations, they give the result that

$$p + \frac{1}{2}\rho v^2 + \rho g z \tag{7}$$

is constant along a streamline. This theorem due to Daniel Bernoulli shows that the pressure drops when the flow velocity increases.

The Venturi meter measures fluid flow by the pressure drop at a constriction. To get water to flow out at high velocity from the narrow nozzle of a hosepipe the water has to be at high pressure behind the nozzle, and the momentum imparted to the water as it leaves causes the hose to push backwards. A plane wing has to be designed to have a faster flow past its upper surface than its lower surface, so that the lower pressure on the top side gives a lift to the wings.

When viscosity is taken into account, additional pressure is needed to drive the fluid against the viscous force, so the quantity in Eq. (7) is not constant, but drops steadily along the tube.

Although viscosity is a mechanism by which ordered kinetic energy of flow can be directly transformed to disordered heat energy, it is only for rather low velocities of flow that it is the dominant mechanism for energy loss. At moderate flow velocities *turbulence* may be set up, in which some of the kinetic energy is converted to swirling motion, vortices, which can usually be seen in fast slowing streams. At even higher speeds, close to the speed of sound, a lot of energy is lost as sound unless great care is taken to minimize such losses. For viscous losses, the drag force increases linearly with speed, but for the other sources of loss the drag usually increases more rapidly with speed.

1.9 Solving hydrodynamic equations

The equations of fluid dynamics are rich in content, but there are few problems that can be solved explicitly in closed form. One example of a problem that can be solved is the flow of fluid down a uniform cylindrical pipe of radius R. Poiseuille's solution of this problem gives a flow parallel to the axis of the pipe, with a speed

$$v_z = v_0(1 - \frac{r^2}{R^2}) \tag{8}$$

that has its maximum v_0 on the axis and falls to zero at the boundary. The viscous force per unit area is

$$f_z = \eta \frac{dv_z}{dr} = -2\eta v_0 \frac{r}{R^2} , \tag{9}$$

where η is the *coefficient of viscosity.* This gives a force on a cylinder of fluid of radius r and length L equal to

$$F_z = 2\pi r L f_z = -4\pi\eta v_0 \frac{r^2}{R^2} L . \tag{10}$$

This is balanced by the pressure gradient, acting on the area πr^2,

$$\frac{dP(z)}{dz} = -4\eta v_0/R^2 . \tag{11}$$

Since the rate of flow through the pipe is proportional to $v_0 R^2$, one can see that the pressure gradient needed to sustain a given rate of flow through a pipe is proportional to η/R^4.

A similar formula can be found for the flow of a viscous fluid between two parallel plates. Gabriel Stokes solved the problem of viscous flow at low speed past a sphere, and found that the viscous drag was $6\pi\eta R v$, where R is the radius of the sphere. This is important in the history of physics, since it was used in Millikan's oil-drop experiment, where the electron charge was determined by balancing electrostatic, gravitational, and viscous forces on a charged drop of oil.

A completely different class of solutions of the equations of motion involve *irrotational flow.* These are solutions in which the velocity can be written as the gradient of a potential function, $\mathbf{v} = \mathbf{grad} S(\mathbf{r})$, just as the electric field in electrostatics or the gravitational force can be written as the gradient of a potential. Such a flow pattern can be used to describe sound waves in a compressible fluid, and, for an incompressible fluid, this gives local solutions of the equations of motion which do not depend on the viscosity. Almost any solution of a problem in electrostatics can therefore be reinterpreted as a possible flow pattern in fluid dynamics. However, most steady flows for real fluids are not irrotational.

Many problems in fluid dynamics, such as the disturbance produced by an object moving with uniform velocity through a fluid, can be broken up into parts where different behavior occurs. Near solid boundaries the flow is usually dominated by viscosity, and solutions can be found numerically, even if analytic solutions are not available. This region is known as the *boundary layer.* Well away from the solid boundaries the flow may well described by smooth and steady flow. There may also be regions where the solution varies with time, perhaps in an irregular manner. Examples of such regions are the wakes left behind a moving sail-boat, or behind the propellers or paddles of a motor boat or canoe. The solution of real problems involves finding appropriate solutions in the different regions, and then matching them together at the interfaces between the different regions.