Moving Non-Relativistic QCD for B Meson Decays

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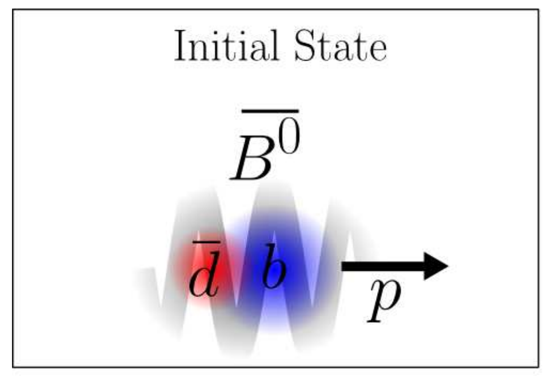
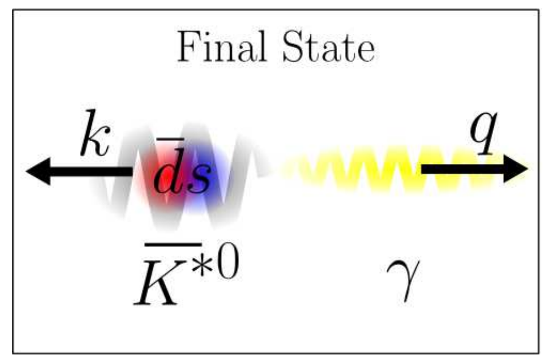
Introduction





To extract CKM matrix elements and test the Standard Model using exclusive electroweak decays of mesons, the form factors have to be known. The short-distance interactions can be calculated perturbatively via an operator product expansion which yields an effective Hamiltonian. The hadronic matrix elements of this Hamiltonian must then be calculated nonperturbatively, and lattice QCD provides a first principles approach to this task.

A very interesting decay, for example, is $B \to K^*\gamma$. As a loop-mediated flavour-changing neutral current process it is particularly sensitive to possible new physics contributions. The accurate lattice calculation of the corresponding form factors is difficult due to the large b quark mass and the high recoil momentum of the K^*, which cause large discretization errors.

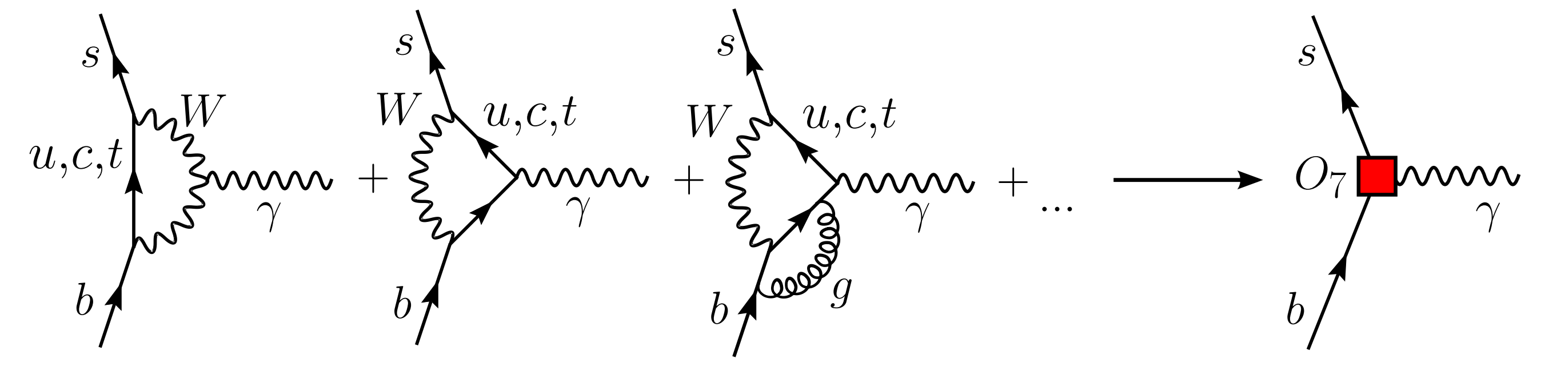
Both problems can be reduced simultaneously by describing the b quark with an effective field theory called *moving Non-Relativistic QCD* (mNRQCD). Here, the B meson is given a large momentum in the opposite direction so that the K^* momentum becomes smaller. The non-relativistic description of the b quark is performed in the reference frame where the B meson is at rest, i.e. in a reference frame moving relative to the lattice. In other words, not only the b quark mass is removed from the effective Lagrangian as in standard NRQCD, but instead a 4-momentum $m_b u$ where u is an arbitrary 4-velocity.

Form Factors and CKM Matrix Elements

The effective weak Hamiltonian relevant to $B \to K^*\gamma$ can be written as

$$\mathcal{H}_{\text{eff}} = -V_{tb}V_{ts}^* \frac{G_F}{\sqrt{2}} \sum_{n=1}^{8} C_n(\mu) O_n,$$

where O_1, ..., O_6 are 4-quark operators, O_7 is an electromagnetic and O_8 is a chromomagnetic dipole operator. The $C_n(\mu)$ are the Wilson coefficients. The operator O_7 for the direct transition $b \to s\gamma$ results from Feynman diagrams like the following,



and is given by

$$O_7 = \frac{e}{16\pi^2} m_b\, T_{\mu\nu} F^{\mu\nu}_{\text{e.m.}} \qquad \text{with} \qquad T_{\mu\nu} = \bar{s}\sigma_{\mu\nu}\frac{1+\gamma_5}{2}b.$$

Neglecting certain long-distance effects, one only needs to calculate the hadronic matrix element of O_7 and hence $T_{\mu\nu}$. It can be parametrized as follows:

$$\begin{aligned} q^\nu \langle K^*(k,\epsilon)|T_{\mu\nu}|B(p)\rangle \;=\; & 2\,T_1(q^2)\,\epsilon_{\mu\nu\rho\sigma}\;\epsilon^{*\nu}\,p^\rho\,k^\sigma \\ & +i\,T_2(q^2)\;\left[\epsilon^*_\mu\,(m_B^2 - m_{K^*}^2) - (\epsilon^*\cdot q)(p+k)_\mu\right] \\ & +i\,T_3(q^2)\,(\epsilon^*\cdot q)\left[q_\mu - \frac{q^2}{m_B^2 - m_{K^*}^2}(p+k)_\mu\right] \end{aligned}$$

with $q \equiv p - k$. For the physical on-shell photon, one has $q^2 = 0$. Note that $T_1(0) = T_2(0)$. The decay rate is

$$\Gamma(B \to K^*\gamma) = \frac{\alpha\, G_F^2}{128\pi^4} C_7(m_b)^2 |V_{tb}V_{ts}^*|^2 m_b^2 m_B^3 \left(1 - \frac{m_{K^*}^2}{m_B^2}\right)^3 |T_1(0)|^2.$$

Form Factors from Correlation Functions

The matrix element $\langle K^*(k,\epsilon)|T_{\mu\nu}|B(p)\rangle$ can be calculated on the lattice from the Euclidean 3-point function

$$C^{(3)}_{\sigma\mu\nu}(\mathbf{k},\,\mathbf{p},\,\tau,\tau') = \sum_{\mathbf{x}}\sum_{\mathbf{x}'}\left\langle \Phi_{K^*_\sigma}(0)\,T^{\text{lat.}}_{\mu\nu}(\mathbf{x},-\tau)\,\Phi^\dagger_B(\mathbf{x}',-\tau')\right\rangle e^{i\mathbf{p}\cdot\mathbf{x}'}e^{-i\mathbf{q}\cdot\mathbf{x}},$$

which for $\tau' \gg \tau \gg 0$ becomes

$$\longrightarrow \sum_\lambda \frac{\sqrt{Z_{K^*}}}{2E_{K^*}} e^{-E_{K^*}\,\tau}\,\frac{\sqrt{Z_B}}{2E_B} e^{-E_B\,(\tau'-\tau)}\,\epsilon^*_\sigma(k,\lambda)\,\langle K^*\,(k,\epsilon(k,\lambda))\,|T_{\mu\nu}|B(p)\rangle.$$

The required amplitudes Z_{K^*}, Z_B and the energies E_{K^*}, E_B can be obtained from the 2-point functions

$$\begin{aligned} C^{(2)}_{\rho\sigma}(\mathbf{k},\,\tau) &= \sum_{\mathbf{x}}\left\langle \Phi_{K^*_\rho}(\mathbf{x},\,\tau)\,\Phi^\dagger_{K^*_\sigma}(0)\right\rangle e^{-i\mathbf{k}\cdot\mathbf{x}} &&\longrightarrow \sum_\lambda \frac{Z_{K^*}}{2E_{K^*}} e^{-E_{K^*}\,\tau}\,\epsilon^*_\rho(k,\lambda)\epsilon_\sigma(k,\lambda), \\ C^{(2)}(\mathbf{p},\,\tau) &= \sum_{\mathbf{x}}\left\langle \Phi_B(\mathbf{x},\tau)\,\Phi^\dagger_B(0)\right\rangle e^{-i\mathbf{p}\cdot\mathbf{x}} &&\longrightarrow \frac{Z_B}{2E_B} e^{-E_B\,\tau}. \end{aligned}$$

Here, $\Phi_{K^*_\sigma}$ and Φ_B are suitable interpolating operators with the structure $\Phi_{K^*_\sigma} \sim \overline{u}\gamma_\sigma s$, $\Phi_B \sim \overline{u}\gamma_5 b$, and $T^{\text{lat.}}_{\mu\nu} = \sum_i C_i T^{(i)\ \text{lat.}}_{\mu\nu}$ is a sum of lattice operators corresponding to the continuum $T_{\mu\nu}$, where the matching coefficients C_i can be calculated perturbatively.

The continuum moving NRQCD Lagrangian

In the calculations of the B meson decay form factors, the heavy b Dirac field $\Psi(x)$ with mass m will be expressed in terms of 2-component mNRQCD quark and antiquark fields $\psi_v(x)$, $\chi_v(x)$, such that a 4-momentum mu is removed from the Lagrangian.

To formally derive the tree-level mNRQCD Lagrangian on Minkowski space, one can start in the reference frame with coordinates x' where the B meson is at rest. The Dirac spinor of the b field in this frame is $\Psi'(x') = [S(\Lambda)]^{-1}\,\Psi(x)$. One then performs a Foldy-Wouthuysen transformation for $\Psi'(x')$,

$$\Psi'(x') = \left[1 + \frac{i\boldsymbol{\gamma}\cdot\mathbf{D}'}{2m} + ...\right] e^{-imx'^0\gamma^0}\;\tilde{\Psi}'(x'),$$

which results in the standard HQET/NRQCD Lagrangian at $\mathbf{v} = 0$,

$$\mathcal{L} = \overline{\tilde{\Psi}'}\left[i\gamma^0 D_0' - \frac{\mathbf{D}^2}{2m} + ...\right]\tilde{\Psi}'.$$

Upon expressing this Lagrangian in the lattice frame (coordinates x), which has a relative 4-velocity u, we obtain

$$\mathcal{L} = \overline{\tilde{\Psi}}\left[i\gamma^0 u\cdot D + \frac{(u\cdot D)^2 - D^2}{2m} + ...\right]\tilde{\Psi},$$

where $\tilde{\Psi}'(x') \equiv \tilde{\Psi}(x)$. This is similar to the HQET Lagrangian with 4-velocity u. It contains higher order time derivatives, which can be removed by a further field redefinition:

$$\tilde{\Psi}(x) = \frac{1}{\sqrt{\gamma}}\left[1 + \frac{i}{4\gamma m}\gamma^0\left[\left(\frac{1}{1-\mathbf{v}^2} - 1\right)D_0 + \left(\frac{1}{1-\mathbf{v}^2} + 1\right)\mathbf{v}\cdot\mathbf{D}\right] + ...\right]\begin{pmatrix}\psi_v \\ \chi_v\end{pmatrix}.$$

This gives $\mathcal{L} = \mathcal{L}_{\psi_v} + \mathcal{L}_{\chi_v}$ with $\mathcal{L}_{\psi_v} = \psi_v^\dagger\,[iD_0 + H]\,\psi_v$, where $H = H_0 + \delta H$ contains no time derivatives. This has the advantage that Green functions can be calculated as an initial value problem. Including terms of order $\mathcal{O}(m^{-2})$, H is given by

$$\begin{aligned} H_0 \;&=\; i\mathbf{v}\cdot\mathbf{D} + \frac{\mathbf{D}^2 - (\mathbf{v}\cdot\mathbf{D})^2}{2\gamma m} \\ \delta H \;&=\; \frac{i\left\{\mathbf{v}\cdot\mathbf{D},\,\mathbf{D}^2\right\} - 2i(\mathbf{v}\cdot\mathbf{D})^3}{4\gamma^2 m^2} + \frac{g\left(\mathbf{D}^{\text{ad}}\cdot\mathbf{E} - \mathbf{v}\cdot(\mathbf{D}^{\text{ad}}\times\mathbf{B})\right)}{8m^2} \\ & \quad + \frac{(2-\mathbf{v}^2)g\left(D_0^{\text{ad}} - \mathbf{v}\cdot\mathbf{D}^{\text{ad}}\right)(\mathbf{v}\cdot\mathbf{E})}{16m^2} + \frac{ig\,\boldsymbol{\sigma}\cdot(\mathbf{D}\times\mathbf{E}' - \mathbf{E}'\times\mathbf{D})}{8\gamma m^2} \\ & \quad - \frac{ig\left\{\mathbf{v}\cdot\mathbf{D},\;\boldsymbol{\sigma}\cdot(\mathbf{v}\times\mathbf{E}')\right\}}{8(\gamma+1)m^2} + \frac{g\boldsymbol{\sigma}\cdot\mathbf{B}'}{2\gamma m} + \frac{ig\left\{\mathbf{v}\cdot\mathbf{D},\;\boldsymbol{\sigma}\cdot\mathbf{B}'\right\}}{4\gamma^2 m^2} \quad + \mathcal{O}(m^{-3}) \end{aligned}$$

with $\mathbf{B}' = \gamma\left(\mathbf{B} - \mathbf{v}\times\mathbf{E} - \frac{\gamma}{1+\gamma}\mathbf{v}(\mathbf{v}\cdot\mathbf{B})\right)$, $\mathbf{E}' = \gamma\left(\mathbf{E} + \mathbf{v}\times\mathbf{B} - \frac{\gamma}{1+\gamma}\mathbf{v}(\mathbf{v}\cdot\mathbf{E})\right)$.

As an effective field theory, mNRQCD is non-renormalisable and requires a UV cut-off lower than m, which will be provided by the lattice. One can not take the continuum limit, but one can systematically include higher order terms in the lattice action (including the light quark and gluon actions) in order to correct for lattice spacing errors.

mNRQCD Green Functions on the Lattice

By definition, the Euclidean Green function $G_{\psi_v}(\mathbf{x},\tau,\mathbf{x}',\tau')$ satisfies for $\tau > \tau'$

$$(D_4 + H)\,G_{\psi_v}(\mathbf{x},\tau,\mathbf{x}',\tau') = 0 \quad\Rightarrow\quad \partial_4 G_{\psi_v}(\mathbf{x},\tau,\mathbf{x}',\tau') = -(H + ig\,A_4)G_{\psi_v}(\mathbf{x},\tau,\mathbf{x}',\tau').$$

Since H does not contain time derivatives, this can be integrated to

$$G_{\psi_v}(\mathbf{x},\tau_2,\mathbf{x}',\tau') = \mathscr{T}\exp\left(-\int_{\tau_1}^{\tau_2}(H + ig\,A_4)\;\mathrm{d}\tau\right)G_{\psi_v}(\mathbf{x},\tau_1,\mathbf{x}',\tau').$$

On the lattice, this evolution equation is conveniently approximated by

$$\begin{aligned} G_{\psi_v}(\mathbf{x},\tau,\mathbf{x}',\tau') \;=\; & \left(1 - \frac{\delta H|_\tau}{2}\right)\left(1 - \frac{H_0|_\tau}{2n}\right)^n U_4^\dagger(\mathbf{x},\tau-1) \\ \times\; & \left(1 - \frac{H_0|_{\tau-1}}{2n}\right)^n\left(1 - \frac{\delta H|_{\tau-1}}{2}\right)G_{\psi_v}(\mathbf{x},\tau-1,\mathbf{x}',\tau'), \end{aligned}$$

where H_0 and H are now discretized. Note that using this equation, Green functions can be calculated in a single pass through the lattice. Antiquark Green functions can be obtained from the complex conjugate of quark Green functions at the negative velocity.

Codes for numerical Simulations

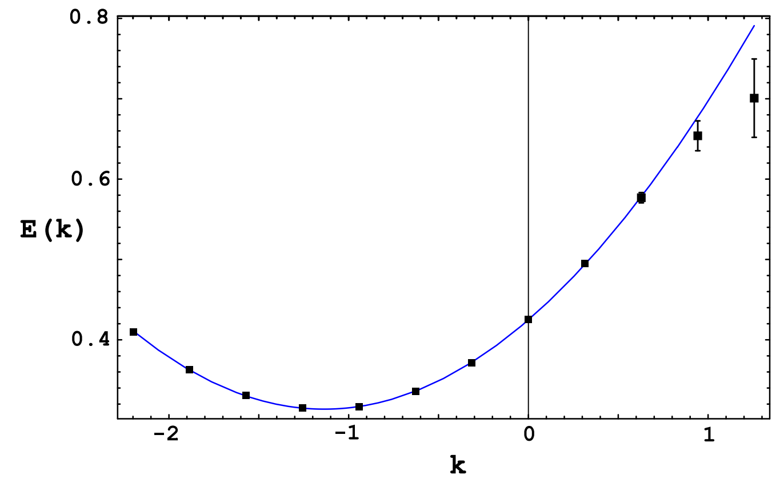
We are currently using a tadpole- and $\mathcal{O}(a^4)$-improved lattice version of the above $\mathcal{O}(\Lambda^2/m^2)$ Lagrangian for simulations. To this end, two codes have been developed:

- An object-oriented C++ code for the automatic generation and simplification of explicit algebraic expressions for improved lattice derivatives. It can output LaTeX and Fortran 90 source code.
- A Fortran 90 code for the calculation of moving NRQCD Green functions on MILC gauge configurations, incorporating the results of the C++ code.

Work in Progress

In order to test the moving NRQCD formalism, we are currently performing simulations for heavy-heavy mesons on MILC gauge configurations. This includes the calculation of dispersion relations for different states of the bottomonium system and decay constants at different velocities. We examine the dependence on the bare parameters m, $\mathbf{v}$, and others.

The renormalization of these parameters is also compared to perturbative results at 1 loop, which have been obtained by L. Khomskii.



Dispersion relation $E(\mathbf{k})$ for the $\eta_b(1S)$ meson at $v = 0.2$. The continuous curve shows a fit $E(\mathbf{k}) = \Delta E_0 + \sqrt{(Z_P\mathbf{P}_0 + \mathbf{k})^2 + M^2_{\text{kin}}}$

Future Plans

The next steps will include tests of moving NRQCD for heavy-light mesons, probably using improved staggered light quarks, and then the calculation of form factors for decays like $B \to \pi l\nu$ and $B \to K^*\gamma$. This will also require perturbative matching calculations for the vector-, axial vector- and tensor currents in mNRQCD.

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