Physics 518: FINAL EXAM

Results you may find useful

Commutation relations:

$$[x_j, p_k] = i\hbar\delta_{jk}\,.$$

Angular momentum:

$$[J_j, J_k] = i\hbar\epsilon_{jk\ell}J_\ell\,, \quad [\vec{J}^2, J_k] = 0\,.$$

$$J_z|j,m\rangle = |j,m\rangle\hbar m\,, \quad J_\pm|j,m\rangle = (J_x \pm iJ_y)|j,m\rangle = |j,m\pm 1\rangle\,\hbar\sqrt{j(j+1) - m(m\pm 1)}\,.$$

Orbital angular momentum $\vec{L} = \vec{x} \times \vec{p}$:

$$\langle\vec{x}'|L_z|\alpha\rangle = -i\hbar\frac{\partial}{\partial\phi}\langle\vec{x}'|\alpha\rangle\,, \quad \langle\vec{x}'|L_\pm|\alpha\rangle = -i\hbar e^{\pm i\phi}\left(\pm i\frac{\partial}{\partial\theta} - \cot\theta\frac{\partial}{\partial\phi}\right)\langle\vec{x}'|\alpha\rangle\,.$$

Rotations. $R_\theta^{\hat{n}}$ is the rotation matrix for a rotation about the axis $\hat{n}$ with angle θ:

$$R_\theta^{\hat{n}} = \exp(-i\theta\hat{n}\cdot\vec{T}) \quad \Rightarrow \quad \mathcal{D}(R_\theta^{\hat{n}}) = \exp(-i\theta\hat{n}\cdot\vec{J}/\hbar)\,.$$

If $\vec{V}$ is a vector operator then $\mathcal{D}(R)^\dagger V_i \mathcal{D}(R) = R_{ij}V_j\,, \quad \Leftrightarrow \quad [J_j, V_k] = i\,\epsilon_{jk\ell}V_\ell\,.$

Spherical tensor operator defined by

$$\mathcal{D}(R)T_q^{(k)}\mathcal{D}^\dagger(R) = \sum_{q'} T_{q'}^{(k)}\mathcal{D}_{q'q}^{(k)}(R)\,,$$

or equivalently

$$[J_z, T_q^{(k)}] = \hbar q T_q^{(k)}\,, \qquad [J_\pm, T_q^{(k)}] = \hbar\sqrt{k(k+1) - q(q\pm 1)}T_{q\pm 1}^{(k)}\,.$$

Relation between vector operator in Cartesian and spherical tensor bases:

$$V_{\pm 1}^{(1)} = \mp\frac{(V_x \pm i\,V_y)}{\sqrt{2}}\,, \quad V_0^{(1)} = V_z\,.$$

Clebsch-Gordon coefficients. Let $\vec{J} = \vec{J}_1 + \vec{J}_2$, with $[J_{1i}, J_{2j}] = 0$. Then

$$|j_1 j_2;\, jm\rangle = \sum_{m_1 m_2} |j_1 j_2;\, m_1 m_2\rangle\langle j_1 j_2;\, m_1 m_2|jm\rangle\,.$$

Wigner-Eckart theorem

$$\langle\alpha'; j'm'|T_q^{(k)}|\alpha; jm\rangle = \langle jk; mq|j'm'\rangle\frac{\langle\alpha' j'||T^{(k)}||\alpha j\rangle}{\sqrt{2j+1}}\,.$$

Reduced mass:

$$1/m_{\rm red} = 1/m_1 + 1/m_2\,.$$

Radial Schrödinger equation

$$\left[\frac{-\hbar^2}{2m}\frac{d^2}{dr^2} + \frac{\hbar^2\ell(\ell+1)}{2mr^2} + V(r) - E\right]u_\ell(r) = 0\,.$$

Hydrogen atom:

$$E_n = -\frac{Ry}{n^2}\,, \quad Ry = \frac{e^2}{2a_0}\,, \quad a_0 = \frac{\hbar^2}{me^2}\,, \quad \alpha = \frac{e^2}{\hbar c}\,.$$

Parity operator Π:

$$\Pi\,\vec{x}\,\Pi = -\vec{x}\,, \quad \Pi\,\vec{p}\,\Pi = -\vec{p}\,, \quad \Pi\,\vec{J}\,\Pi = \vec{J}\,, \quad \Pi^2 = \mathbf{1}\,.$$

Time-reversal operator θ:

$$\theta\,\vec{x}\,\theta^{-1} = \vec{x}\,, \quad \theta\,\vec{p}\,\theta^{-1} = -\vec{p}\,, \quad \theta\,\vec{J}\,\theta^{-1} = -\vec{J}\,, \quad \theta^2 = \pm\mathbf{1}$$

θ is anti-unitary:

$$\theta(c_1|\alpha\rangle + c_2|\beta\rangle) = c_1^\star\theta|\alpha\rangle + c_2^\star\theta|\beta\rangle\,,$$

$$\text{If } |\widetilde{\alpha}\rangle = \theta|\alpha\rangle \text{ and } |\widetilde{\beta}\rangle = \theta|\beta\rangle \text{ then } \langle\widetilde{\alpha}|\widetilde{\beta}\rangle = \langle\alpha|\beta\rangle^\star\,.$$

Non-degenerate perturbation theory

$$\begin{aligned} E_n - E_n^{(0)} &= \langle n^{(0)}|V|n^{(0)}\rangle + \langle n^{(0)}|V \left[\frac{1 - |n^{(0)}\rangle\langle n^{(0)}|}{E_n^{(0)} - H_0}\right] V\,|n^{(0)}\rangle + O(V^3) \\ |n\rangle &= |n^{(0)}\rangle + \left[\frac{1 - |n^{(0)}\rangle\langle n^{(0)}|}{E_n^{(0)} - H_0}\right] V|n^{(0)}\rangle + O(V^2) \end{aligned}$$

Schrödinger time evolution operator:

$$U(t_2, t_1) = T\left\{\exp\left(\frac{1}{i\hbar}\int_{t_1}^{t_2} dt H(t)\right)\right\}$$

Interaction representation (for constant H_0):

$$|\alpha, t\rangle_I = e^{iH_0 t/\hbar}|\alpha, t\rangle_S\,, \qquad |\alpha, t_2\rangle_I = U_I(t_2, t_1)|\alpha, t_1\rangle_I\,,$$

$$V_I(t) = e^{iH_0 t/\hbar}V(t)e^{-iH_0 t/\hbar}\,,$$

$$U_I(t_2, t_1) = T\left\{\exp\left(\frac{1}{i\hbar}\int_{t_1}^{t_2} dt V_I(t)\right)\right\}\,.$$

Dyson series:

$$U_I(t_2, t_1) = \mathbf{1} + \frac{1}{i\hbar}\int_{t_1}^{t_2} dt V_I(t) + O(V_I^2)$$

Fermi's Golden rule (due to part of interaction with form $V\exp(-i\omega t)$). Transition rate from an eigenstate of H_0, $|i\rangle$, having energy E_i, to a continuum of final states is

$$w_{i\to[f]} = \frac{2\pi}{\hbar}|\langle f|V|i\rangle|^2\rho(E_f)\,,$$

where $E_f = E_i + \hbar\omega$, and it is assumed that the final states all have the same matrix element of V (otherwise an average will be needed).