Physics 505: CLASSICAL MECHANICS

FINAL EXAMINATION: Results you may find useful

Kinematics

Cylindrical coordinates in three dimensions:

$$x = r\,\cos\phi\,,\quad y = r\,\sin\phi\,,\quad \dot{\vec{r}} = \dot{r}\,\hat{r} + r\dot{\phi}\,\hat{\phi} + \dot{z}\,\hat{z}\,;$$

Spherical coordinates in three dimensions:

$$x = r\,\sin\theta\cos\phi\,,\quad y = r\,\sin\theta\sin\phi\,,\; z = r\,\cos\theta\,,\qquad \dot{\vec{r}} = \dot{r}\,\hat{r} + r\dot{\theta}\,\hat{\theta} + r\sin\theta\,\dot{\phi}\,\hat{\phi}\,;$$

Basic dynamics

Motion in inertial frame:

$$\dot{\vec{P}}_{CM} = \vec{F}^{\rm ext,tot}\,,\quad \vec{L}_{CM} = \vec{R}_{CM}\times\vec{P}_{CM}\,,\quad \dot{\vec{L}}_{CM} = \vec{R}_{CM}\times\vec{F}^{\rm ext,tot}\,;\qquad \dot{\vec{L}} = \vec{\Gamma}_{\rm ext,tot}$$

Angular momentum relative to CM ($\vec{r}\,'_i = \vec{r}_i - \vec{R}_{CM}$, etc.)

$$\vec{L}\,' = \vec{L} - L_{CM} = \sum_i \vec{r}\,'_i\times\vec{p}\,'_i\,,\quad \dot{\vec{L}}\,' = \sum_i \vec{r}\,'_i\times\vec{F}_i^{\rm ext}\,;$$

Reduced mass: $\mu = m_1 m_2/(m_1+m_2)$

Orbits and scattering

Effective central potential:

$$V_{\rm eff}(r) = V(r) + \frac{\ell^2}{2\mu r^2}\,,\qquad \frac{d\phi}{dt} = \frac{\ell}{mr^2}\,;$$

Ellipse and hyperbola (focal length f, semi-major axis a, eccentricity $\epsilon = f/a$):

$$r(1-\epsilon\cos\phi) = a(1-\epsilon^2)$$

Differential cross section

$$\frac{d\sigma}{d\Omega} = \frac{b}{\sin\theta}\left|\frac{db}{d\theta}\right|$$

Rotating frame

Newton's law in a rotating, accelerating frame ($\vec{r}_{NI}$ is the position vector *in the rotating frame*; $\vec{r}_0(t)$ is the position of the origin of the rotating frame in the inertial frame; $\omega(t)$ is the angular velocity of the rotating frame about this origin):

$$m\,\ddot{\vec{r}}_{NI} = \vec{F} - m\,\ddot{\vec{r}}_0 - 2m\,\vec{\omega}\times\dot{\vec{r}}_{NI} - m\,\vec{\omega}\times(\vec{\omega}\times\vec{r}_{NI}) - m\,\dot{\vec{\omega}}\times\vec{r}_{NI}\,;$$

Lagrangian dynamics

Lagrangian and conjugate momenta:

$$L(\{q_\alpha\},\{\dot q_\alpha\},t) = T - V\,; \qquad p_\alpha \equiv \frac{\partial L}{\partial \dot q_\alpha}\,;$$

Hamilton's principle:

$$\delta S = \delta \int_{t_1}^{t_2} dt L = 0\,; \tag{1}$$

Holonomic constraints (using generalized coordinates q_α):

$$f_j(\{q_\alpha\},t) = 0\,, \qquad j = 1, \text{number of constraints}\,;$$

Implement using Lagrange multipliers:

$$L \longrightarrow L + \sum_j \lambda_j f_j\,.$$

Euler-Lagrange equation with constraints (if present) implemented by Lagrange multipliers:

$$\frac{d}{dt}\frac{\partial L}{\partial \dot q_\alpha} - \frac{\partial L}{\partial q_\alpha} = \sum_j \lambda_j \frac{\partial f_j}{\partial q_\alpha} = F_\alpha^{\text{gen}}\,, \tag{2}$$

where F_α^{gen} is the generalized force of constraint in the q_α direction, which does virtual work $\delta W = F_\alpha^{\text{gen}}\delta q_\alpha$ when q_α is varied.

<u>Small oscillations and normal modes</u> (η a vector, $\mathbf{M}$, $\mathbf{v}$ matrices):

$$\eta_\alpha = \mathbf{q}_\alpha - \mathbf{q}_\alpha^{\mathbf{0}}\,, \quad \mathbf{M}_{\alpha\beta} = \sum_{\mathbf{i}} \mathbf{m_i} \left(\frac{\partial \mathbf{x_i}}{\partial \mathbf{q}_\alpha}\right)_{\mathbf{q^0}} \left(\frac{\partial \mathbf{x_i}}{\partial \mathbf{q}_\beta}\right)_{\mathbf{q^0}}\,, \quad \mathbf{v}_{\alpha\beta} = \left(\frac{\partial^2 \mathbf{V}}{\partial \mathbf{q}_\alpha \partial \mathbf{q}_\beta}\right)_{\mathbf{q^0}}$$

$$L = \frac{1}{2}\dot\eta^{\mathbf{T}} \cdot \mathbf{M} \cdot \dot\eta - \frac{1}{2}\eta^{\mathbf{T}} \cdot \mathbf{v} \cdot \eta + \mathbf{O}(\eta^{\mathbf{3}})$$

$$\left(\mathbf{v} - \omega_s^2 \mathbf{M}\right)\rho^{(\mathbf{s})} = 0\,, \qquad {\rho^{(\mathbf{s})}}^T \cdot \mathbf{M} \cdot \rho^{(\mathbf{t})} = \delta^{st}\,, \qquad {\rho^{(\mathbf{s})}}^T \cdot \mathbf{v} \cdot \rho^{(\mathbf{t})} = \omega_s^2\,\delta^{st}\,.$$

Modal matrix $\mathbf{A}$ and normal coordinates ζ:

$$\mathbf{A}_{\alpha\beta} = \rho_\alpha^{(\beta)}\,, \qquad \zeta \equiv \mathbf{A^T} \cdot \mathbf{M} \cdot \eta\,, \qquad \eta = \mathbf{A} \cdot \zeta$$

<u>Hamiltonian dynamics</u>

Hamiltonian and Hamilton's equations:

$$H(p,q,t) = \sum_\alpha p_\alpha \dot q_\alpha - L\,, \qquad \frac{\partial H}{\partial p_\alpha} = \dot q_\alpha\,, \quad \frac{\partial H}{\partial q_\alpha} = -\dot p_\alpha\,, \quad \frac{dH}{dt} = \frac{\partial H}{\partial t} = -\frac{\partial L}{\partial t}\,;$$

If the potential depends on q_α, but not on $\dot q_\alpha$ or t, and if the generalized coordinates are time independent, then $H = T + V$.

Canonical tranformations with generating function $F(q,Q,t)$:

$$\begin{aligned} \widetilde{H}(P,Q,t) &= H(p,q,t) + \frac{\partial}{\partial t}F(q,Q,t)\,, \\ p_\alpha &= \frac{\partial}{\partial q_\alpha}F\,, \qquad P_\alpha = -\frac{\partial}{\partial Q_\alpha}F\,, \end{aligned}$$

and with generating function $S(q,P,t) = F + \sum_\alpha P_\alpha Q_\alpha$:

$$\begin{aligned} \widetilde{H}(P,Q,t) &= H(p,q,t) + \frac{\partial}{\partial t} S(q,P,t)\,, \\ p_\alpha &= \frac{\partial}{\partial q_\alpha} S\,, \qquad Q_\alpha = \frac{\partial}{\partial P_\alpha} S\,; \end{aligned}$$

$\widetilde{H} = 0$ if $S(q,P,t)$ satisfies Hamilton-Jacobi equation:

$$H(\left\{\frac{\partial S}{\partial q_\alpha}\right\}, \{q_\alpha\}, t) + \frac{\partial S}{\partial t} = 0\,;$$

If H is independent of t, then can separate this equation using

$$S(q,P,t) = W(q,P) - Et\,.$$

Poisson brackets:

$$[F,G]_{\mathrm{PB}} \equiv \sum_\alpha \frac{\partial F}{\partial q_\alpha}\frac{\partial G}{\partial p_\alpha} - \frac{\partial F}{\partial p_\alpha}\frac{\partial G}{\partial q_\alpha}\,,$$

with F and G functions of phase-space and t. Hamilton's equations are then equivalent to

$$\frac{dF}{dt} = \frac{\partial F}{\partial t} - [H,F]_{\mathrm{PB}}\ .$$

A simple test for whether a transformation $(p,q) \to (P,Q)$ is canonical is whether

$$[P_\alpha, Q_\beta]_{\mathrm{PB}} = -\delta_{\alpha,\beta}\,, \text{ and } [Q_\alpha, Q_\beta]_{\mathrm{PB}} = [P_\alpha, P_\beta]_{\mathrm{PB}} = 0\,.$$

Rigid body motion

Inertia tensor:

$$\mathbf{I}_{ij} = \int d^3r \rho(\vec{r}) \left(\delta_{ij} r^2 - x_i x_j\right)$$

is diagonalized ($\mathbf{I} = \mathrm{diag}(I_1, I_2, I_3)$) if Cartesian basis chosen along principal axes.
Angular momentum and kinetic energy of rigid body:

$$\vec{L} = \sum_{j=1}^{3} I_j\,\omega_j\,\hat{e}_j\,, \qquad T = \sum_{j=1}^{3} \frac{1}{2} I_j\,\omega_j^2 = \frac{1}{2}\vec{L}\cdot\vec{\omega}\,;$$

Euler's equations in body-fixed frame ($\vec{\tau}$ is torque about CM [if body is rotating freely] or about a point of the rotating body fixed in an inertial frame):

$$I_1\,\dot{\omega}_1 - (I_2 - I_3)\omega_2\,\omega_3 = \tau_1\,, \quad \text{plus cyclic permutations}$$

Angular velocity of rigid body in body-fixed coordinates defined by Euler angles:

$$\begin{aligned} \omega_1 &= -\dot{\alpha}\,\sin\beta\,\cos\gamma + \dot{\beta}\,\sin\gamma\,, \\ \omega_2 &= \dot{\alpha}\,\sin\beta\,\sin\gamma + \dot{\beta}\,\cos\gamma\,, \\ \omega_3 &= \dot{\alpha}\cos\beta + \dot{\gamma}\,; \end{aligned}$$

Chaotic dynamics

Lyapunov exponent for the map $x_{n+1} = f(x_n)$:

$$\lambda = \lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} \ln |f'(x_j)| \, ;$$

Capacity dimension if $N(a)$ hyperspheres of radius a are required to cover the object (a_0 being a fixed reference length):

$$d_C = \lim_{a\to 0} \frac{\ln N(a)}{\ln(a_0/a)} \, ;$$