

Adiabatic Theorem and Quantum Hall Effect

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1 Introduction

The Adiabatic Theorem is a result in quantum mechanics. Its statement depends on the conditions on the Hamiltonian. The earliest result, proved in 1928 by Born, concerns bounded Hamiltonians with discrete eigenvalues, with non-degenerate initial eigenstate. In 1950 Kato extended the theorem to Hamiltonians with some continuous spectra. Since then more generalizations have been given. Throughout the paper, we will only work with bounded Hamiltonians with discrete spectra.

2 Adiabatic Theorem

2.1 General Idea

Consider a time dependent Hamiltonian $H(t)$. As H evolves with time, so do its eigenvalues and the associated eigenspaces. Suppose H varies slowly (i.t. dH/dt is small), then an eigenstate with eigenvalue E_1 at $t=0$ will evolve in such a way that at time t , it is still an eigenstate with eigenvalue $E_1(t)$.

2.2 Basic Formulation

The basic adiabatic theorem concerns a Hamiltonian with discrete eigenvalues, each having non-degenerate eigenstates. Suppose $H(t)$ is such a Hamiltonian, with eigenvalues $E_n(t)$ and eigenfunctions $\phi_n(t)$. At any instant t , $H(t)\phi_n(t) = E_n(t)\phi_n(t)$.

Let $\Phi(t)$ be the general solution to the Schroedinger equation

$$i\hbar \frac{\partial}{\partial t} \Phi(t) = H(t)\Phi(t) \quad (1)$$

Then we can always write $\Phi(t)$ as the linear combination of eigenfunctions: $\Phi(t) = \sum_n C_n(t)\phi_n(t)e^{i\theta_n(t)}$. The expression $\theta_n(t)$ is a phase factor given by $\theta_n(t) = -\frac{1}{\hbar} \int_0^t E_n(t') dt'$, and it is called the dynamic phase.

In particular, at $t=0$, $\Phi(t) = \sum_n C_n(0)\phi_n(0)e^{i\theta_n(0)}$.

By substituting $\Phi(t)$ into the Schroedinger equation and other mathematical manipulations [1], we can show that

$$\dot{C}_n = -C_n \langle \phi_n | \dot{\phi}_n \rangle - \sum_{n \neq m} C_m \frac{\langle \phi_n | \dot{H} | \phi_m \rangle}{E_m - E_n} e^{i\alpha}.$$

The second sum is small if \dot{H} is small. In the adiabatic limit, $\dot{H} \rightarrow 0$, so we can drop the second sum. The solution to the resulting differential equation is

$$C_n(t) = C_n(0)e^{i\gamma_n(t)} \quad (2)$$

In other words, the coefficient at time t is just the coefficient at $t=0$ multiplied by a phase factor. Accordingly,

$$\Phi(t) = e^{i\theta_n(t)} e^{i\gamma_n(t)} \phi_n(t) \quad (3)$$

In particular, if $\Phi(t)$ started out as the m^{th} eigenstate, i.e. $\Phi(0) = \phi_m(0)$, then the coefficient $C_n(0)$ is just δ_{nm} . Consequently, $C_n(t)$ is δ_{nm} multiplied by a phase factor, which means

$$\Phi(t) = e^{i\gamma_m(t)} e^{i\theta_m(t)} \phi_m(t) \quad (4)$$

This is the statement that $\Phi(t)$ remains the m^{th} eigenstate, multiplied by two phase factors. We now have the adiabatic theorem.

The phase factor γ is called the geometric phase. It arises from the transport of the Hamiltonian in the parameter space (which is another way to formulate the time-dependence of the Hamiltonian). If the Hamiltonian returns to its initial state (i.e. it is transported through a closed loop), then the phase factor is called Berry's phase.

2.3 Degenerate Eigenstates

In the previous section we assumed that the eigenstates of $H(t)$ are non-degenerate. We will lift that restriction here. If the eigenstates are degenerate, then to each eigenvalue $E_n(t)$ we can associate an eigenspace of dimension m , where m is the multiplicity. In this case, the adiabatic theorem is formulated differently.

First we will introduce the projection operator $P_n(t)$ as the projection onto the n^{th} eigenspace. In other words, for any function Φ , $P_n(t)\Phi$ is the component of Φ in the n^{th} eigenspace. $P_n(t)$ is Hermitian and satisfies $P^2=P$. We can then write the Hamiltonian in the bases of its projection operators:

$$H(t) = \sum_n E_n(t)P_n(t) \quad (5)$$

Secondly, for convenience we will make the change of variable t to $s=t/\tau$, where τ is the time period over which Hamiltonian evolves from initial to final state. s is called the scaled time, and it ranges from 0 to 1. With the new variable, the Schroedinger equation reads

$$i\hbar\partial_s\Phi(s) = \tau H(s)\Phi(s) \quad (6)$$

The condition of the adiabatic theorem, that H changes slowly, can then be stated as $\tau \rightarrow \infty$.

The adiabatic evolution of the system is an approximation of the physical evolution (i.e. how the system actually changes with time). This approximation only works when $\tau \rightarrow \infty$. Before we define the adiabatic evolution, we should consider the physical evolution.

Let $U_\tau(s)$ be an unitary operator that solves the initial value problem

$$i\hbar\partial_s U_\tau(s) = \tau H(s)U_\tau(s), \quad U_\tau(s) = 1 \quad (7)$$

By definition, U_τ satisfies the following property: If $\Phi(s)$ is the solution to the Schroedinger equation at $s=0$, then $U_\tau(s)\Phi(0)$ solves the Schroedinger equation at time s . In other words, U_τ encodes how the physical state of the system evolves. Hence, we will call U_τ the physical evolution operator.

Our next task is to define an adiabatic evolution that approximates the physical evolution. There are two parts to the process.

First, let $U_A(s, P_n)$ be a unitary operator that describes the adiabatic evolution of the n th eigenspace. $U_A(0, P_n) = 1$. Basically, suppose $\phi(0)$ is a wavefunction in the n th eigenspace, then $U_A(s, P_n)\phi(0)$ is the time-evolved function at time s . Moreover, the adiabatic evolution of any wavefunction can be expressed as a linear combination of the evolution of eigenfunctions. For example, suppose $\Phi(0)$ satisfies the Schroedinger equation at time 0. Recall that $P_n(0)\Phi(0)$ is the component of $\Phi(0)$ in the n th eigenspace, so that $\Phi(0) = \sum_n P_n(0)\Phi(0)$. At time s , adiabatic evolution gives $\sum_n U_A(s, P_n)P_n(0)\Phi(0)$. On the other hand, physical evolution gives $\sum_n U_\tau P_n(0)\Phi(0)$. The goal is to have $U_A(s, P_n)$ approximate $U_\tau(s)$. Specifically, we want

$$U_\tau(s) = U_A(s, P_n) + O(1/\tau) \quad (8)$$

in which $O(1/\tau)$ is a quantity of order $1/\tau$.

Let us now clarify what makes $U_A(s, P_n)$ an Adiabatic Evolution. We want U_A to be such that a wavefunction which started out in the n th eigenspace remains in that eigenspace. This will be true if $U_A(s, P_n)$ satisfies the following conditions:

$$U_A(s, P_n)P_n(0) = P_n(s)U_A(s, P_n) \quad (9)$$

$$U_A(s, P_n)(1 - P_n(0)) = (1 - P_n(s))U_A(s, P_n) \quad (10)$$

They are called the decoupling conditions. While $P_n(s)$ represents the projection onto n th eigenspace, $1 - P_n(s)$ is the projection onto the complement of the n th eigenspace. This is analogous to the 2-D decomposition of a vector into parallel component and perpendicular component. $P_n(s)$ and $1 - P_n(s)$ are by definition orthogonal. The decoupling condition further imposes that $U_A(s, P_n)P(0)$ and $U_A(s, 1 - P_n)(1 - P_n(0))$ are orthogonal, which is sufficient to guarantee that eigenfunctions will be confined to their eigenspace. In summary, we say that $U_A(s, P_n)$ maps the n th eigenspace at time $s=0$ onto the eigenspace at time s . Having defined U_A , we will represent the adiabatic evolution of the Hamiltonian in a similar fashion. Consider the n th eigenspace. We will define a Hermitian operator $H_A(s, P_n)$ with the following property:

$$i\hbar\partial_s U_A(s, P_n) = \tau H_A(s, P_n)U_A(s, P_n) \quad (11)$$

Essentially, this requires that for any $\Phi(0)$ in the n th eigenspace, its adiabatic evolution solves a modified Schroedinger equation:

$$i\hbar\partial_s U_A(s, P_n)\Phi(0) = \tau H_A(s, P_n)U_A(s, P_n)\Phi(0) \quad (12)$$

In which $H(t)$ is replaced by a P_n -specific operator. Since U_A is a unitary operator, we have from equation 11 that

$$H_A(s, P_n) = i\hbar/\tau[\partial_s U_A(s, P_n)]U_A^\dagger(s, P_n) \quad (13)$$

Substituting the expression into the decoupling conditions and using the fact that P and $1-P$ are orthogonal, we find that

$$H_A(s, P_n) = H(s) + i/\tau [P'_n(s), P_n(s)] \quad (14)$$

This will be the formal definition of H_A .

The preceding derivations establish the form of the adiabatic evolution, which preserves eigenspaces. The Adiabatic Theorem is just a statement that given certain conditions on $H(t)$, an adiabatic evolution can always be defined which approximates the physical evolution, up to differences of order $1/\tau$. Typically, the conditions on H include that it is bounded, that its domain in the Hilbert space is closed and time-independent, and that it is continuously differentiable [2]. We have also been relying on the condition that H has discrete eigenvalues. The non-discrete case requires a different formulation and will not be discussed here.

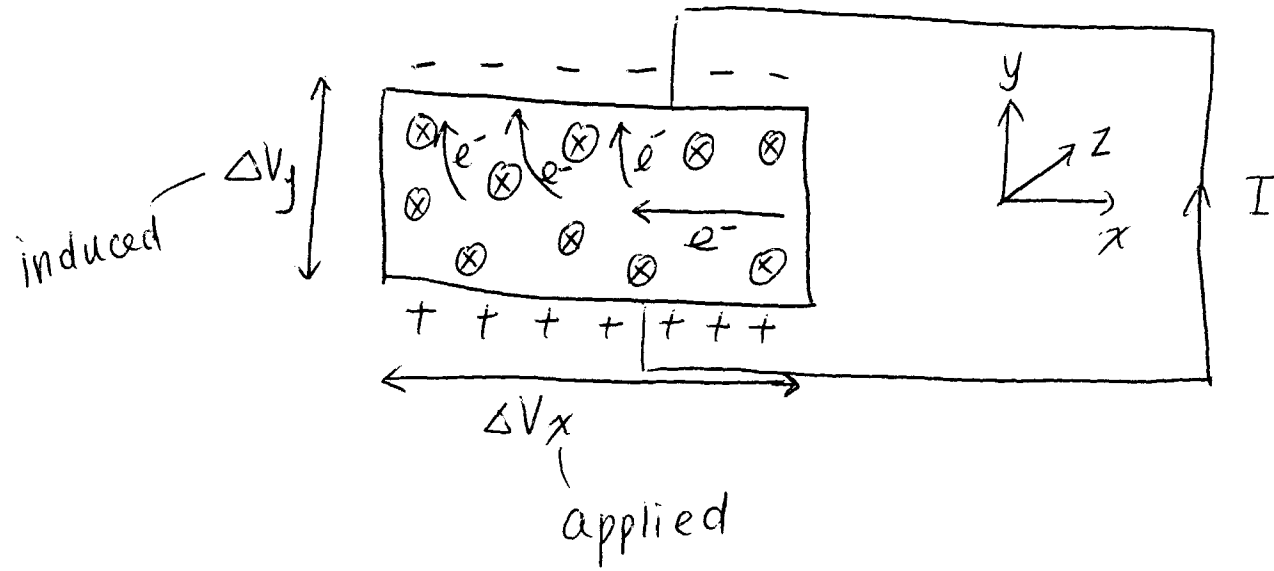
In early proofs of the adiabatic theorem, the Gap Condition is also imposed, which requires finite gaps in the spectrum of H . A more general adiabatic theorem without this condition has since been proved.

3 Quantum Hall Effect

3.1 General Idea

The quantum Hall effect is observed in 2D electron gas, which can be found in such as MOSFET and graphenes. Analogous to the classical Hall effect, the quantum Hall effect describes the apparent potential difference transverse to applied voltage and B field. An illustration is on the next page.

figure 1



In the figure, there's an applied potential difference in the x direction, while the B field points in the z direction. As electrons move against V_x , they are deflected in the y direction by the magnetic field. Consequently, there is a charge separation in the y direction, which then induces V_y . The induced voltage drives current I.

The Hall conductance σ is defined as the ratio I/V_x . The distinct feature of the quantum Hall Effect, as opposed to classical, is that σ is quantized. In cases where the electron-electron interaction is weak, σ is an integer multiple of $e^2/2\pi\hbar$. This is called the integer quantum Hall effect. It arises from the quantization of cyclotron orbits of the electrons in the B field. The Hamiltonian of a single electron has the form

$$H = \frac{p_x^2}{2m} + \frac{1}{2}m\omega_c^2\left(x - \frac{\hbar k_y}{m\omega_c}\right)^2 \quad (15)$$

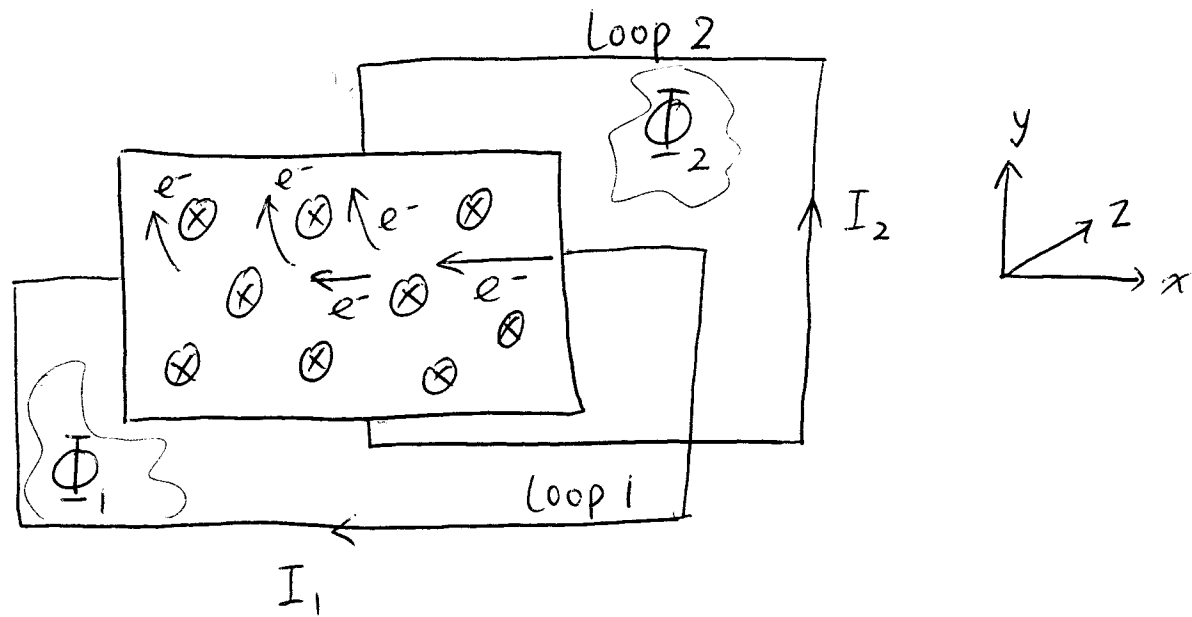
which is similar to that of a 1D harmonic oscillator. Consequently, an electron can occupy the energy levels $E_n = \hbar\omega_c(1/2+n)$. They are called Landau Levels, and they are highly degenerate due to the wave number k_y in the Hamiltonian. In fact, the degeneracy is proportional to B. Thus, with strong magnetic field, electrons can occupy only a few Landau levels. This is where Hall Effect can be observed.

3.2 Apply the Adiabatic Theorem

We will consider the Hamiltonian of the 2D electron gas, instead of a single electron. The energy levels of this Hamiltonian will also be highly degenerate, so we need the formulation given in section 1C.

We will also consider a slightly different setup, which is illustrated on the next page.

Figure 2



Once again, magnetic field B_z is applied on the electron gas. However, there are now two loops connected to the gas. In loop 1, there is a time-dependent magnetic flux Φ_1 . As Φ_1 changes with time, it induces an emf in loop 1, and hence the potential difference V_x on the electron gas. Electrons are driven in the x direction by V_x , deflected in the y direction by B_z , and transported through loop 2. The current I in loop 2 induces Φ_2 . The Hamiltonian of the system is

$$\tilde{H}(\Phi_1, \Phi_2) = \sum_j (1/2(-i\partial_j - \vec{A}(\vec{x}_j) - \Phi_1 \vec{a}_1(\vec{x}_j) - \Phi_2 \vec{a}_2(\vec{x}_j))^{-2} + V(\vec{x}_j))^2 + \sum_{k < j} 1/|\vec{x}_k - \vec{x}_j| \quad (16)$$

where the sum is over all the electrons in the gas, each at position \vec{x}_j . In the equation, \vec{A} is the vector potential associated with the field B_z , \vec{a}_1 is the vector potential associated with Φ_1 , and \vec{a}_2 is the vector potential associated with Φ_2 . Here \tilde{H} depends on two parameters Φ_1 and Φ_2 . Since Φ_1 is a function of time, we can collapse this parameter and let H be dependent on s and the parameter Φ_2 . For convenience we set Φ_2 to equal ϕ .

Define

$$H(s, \phi) = \tilde{H}(\Phi_1, \Phi_2) \quad (17)$$

$$P(s, \phi) = \tilde{P}(\Phi_1, \Phi_2) \quad (18)$$

By collapsing a parameter, we are essentially restricting \tilde{H} and \tilde{P} to the Φ_2 axis on the parameter space. On this axis, Φ_1 equals 0, which is the initial condition of the system. Similarly, we can restrict the adiabatic evolution to the axis Φ_2 by the operator $\tilde{U}_A(\phi)$. This is essentially a parallel transport of eigenspaces along the Φ_2 .

$\tilde{U}_A(\phi)$ is the solution to the initial value problem

$$i\hbar\partial_s\tilde{U}_A(\phi) = \tau\tilde{H}(\phi)\tilde{U}_A(\phi), \quad \tilde{U}_A(0) = 1 \quad (19)$$

where

$$H_A(\phi) = \tilde{H}(0, \phi) + i/\tau[\partial_\phi\tilde{P}(0, \phi), \tilde{P}(0, \phi)] \quad (20)$$

Fix ϕ so that only s varies. Now we can define the adiabatic operators $H_A(s, \phi)$ and $U_A(s, \phi)$ in the usual way.

$$H_A(\phi) = H(s, \phi) + i/\tau[\partial_s P(s, \phi), P(0, \phi)] \quad (21)$$

$$i\hbar\partial_s U_A(s, \phi) = \tau H_A(s, \phi) U_A(s, \phi) \quad (22)$$

with the initial condition that $U_A(0, \phi) = 1$.

The reason we have both $\tilde{U}_A(\phi)$ and $U_A(s, \phi)$ is that we can split the evolution of the system in the Φ_1, Φ_2 parameter space into two steps. Initially, $s=0, \Phi_1=0$, and $\Phi_2=0$, and suppose the system is described by the wavefunction f. To see how f evolves, we first apply $\tilde{U}_A(\phi)$ to f, which transports f along the Φ_2 axis to a certain flux value ϕ . Then we apply $U_A(s, \phi)$, the evolution in time, which is equivalent to transporting f along a curve parallel to the Φ_1 axis. Thus, the evolved wavefunction, as function of s and ϕ , is

$$f(s, \phi) = U_A(s, \phi)\tilde{U}_A(\phi)f \quad (23)$$

3.3 Integer Quantum Hall Effect

We will work in the units $\hbar=1$ and $e=1$, so that the Hall conductance comes out as multiple of $1/2\pi$. We will make use of the magnetic flux quantum, $\Phi_o = \pi\hbar/e$, which is just π in our units.

To show that Hall conductance is a multiple of $1/2\pi$ is equivalent to showing that, as Φ_1 changes by one magnetic flux quantum, the average charge transported in loop 2 is an integer. [2]

Let Φ_1 increase from 0 to one flux quantum as s goes from 0 to 1. During the process, the charge transported through loop 2 is

$$Q_A(\phi) = \tau \int_0^1 I(s, \phi) ds \quad (24)$$

We want to take the average of Q_A over ϕ .

$$\langle Q_A \rangle = \frac{1}{\pi} \int_0^\pi Q_A(\phi) d\phi \quad (25)$$

Finally, we need to write the charge transported in terms of adiabatic wavefunction. With the previously defined $f(s, \phi)$, (equation 23), we can show that [2]

$$I = \frac{i}{\tau} \partial_s \langle f(s, \phi), \partial_\phi f(s, \phi) \rangle \quad (26)$$

$$Q_A(\phi) = i \langle f(1, \phi), \partial_\phi f(1, \phi) \rangle \quad (27)$$

Thus, the average charge transported is $\frac{i}{\pi} \int_0^\pi \langle f(1, \phi), \partial_\phi f(1, \phi) \rangle d\phi$. By the geometric formulation of this problem, $\langle Q_A \rangle$ is a First Chern Character associated with ground state of $\tilde{H}(\Phi_1, \Phi_2)$ [2]. Hence it is an integer. Consequently, the Hall conductance is an integer multiple of $1/2\pi$.

References

- [1] Griffiths. Introduction to Quantum Mechanics.
- [2] Avron et al. Adiabatic Theorems and Applications to the Quantum Hall Effect. Communications in Mathematical Physics. 1987.