The Casimir Effect

Kevin Bell

In 1948, Casimir and Dover published a paper addressing a disparity between the conventional theory of the Van der Waals-London force and the observed interaction of colloidal particles. The paper was called *The Influence of Retardation on the London-Van Der Waals Forces*, and, as the name implies, it discussed how the fact that electric field information travels only at the speed of light. In the process, Casimir considered the interaction between a neutral atom and a grounded plate [1]. It was this thought experiment that motivated another paper published in the same year by Casimir alone entitled *On the attraction between two perfectly conducting plates*. This little paper, all of two and a half pages long, concluded that there is an attractive force exerted between two nearby conducting plates in order to minimize the so-called vacuum state energy [2]. The effect was later confirmed experimentally, and has since proven to be of significance in other areas of science, from the ultra-small technology of MEMS, to the fundamental nature of space and the universe itself.

Canonical Quantization in Brief

Classically, the energy in the electromagnetic fields is

$$u = \varepsilon_0 E^2 + \frac{1}{\mu_0} B^2 dV$$

If we represent the electric and magnetic fields in a one-dimensional box as

$$E_y = p(t) D \sin(kx)$$

$$B_z = q(t) \frac{D}{c} \cos(kx)$$

Where p and q are arbitrary functions of time and D is a constant. In order for these expressions to satisfy Maxwell's equations, we must have

$$\frac{dq}{dt} = \omega p \quad \text{and} \quad \frac{dp}{dt} = -\omega q$$

The Hamiltonian per unit area in the box is

$$H = \int_0^L u dx = \frac{D^2}{2} \int_0^L \left(\varepsilon_0 p^2 \sin^2 kx + \frac{1}{\mu_0 c^2} q^2 \cos^2 kx \right) dx = \frac{D^2 L \varepsilon_0}{4} \left(p^2 + q^2 \right)$$

if we now choose $D = \sqrt{\frac{2\omega}{L\varepsilon_0}}$ so that

$$H = \frac{\omega}{2}(p^2 + q^2)$$

then p and q can be seen to be perfectly analogous to momentum and position respectively, as Hamilton's equations are satisfied by

$$\frac{\partial H}{\partial p} = \omega p = \frac{dq}{dt} \quad \text{and} \quad \frac{\partial H}{\partial q} = \omega q = -\frac{dp}{dt}$$

The Hamiltonian above is already very similar to that of the quantum harmonic oscillator. Simply replacing the momentum above with the quantum mechanical momentum operator and making a trivial change to dimensionless units completes the similarity. One can then show that, completely analogously to the quantum harmonic oscillator, $e_n = \hbar\omega\left(n + \frac{1}{2}\right)$, where n now represents the number of photons present in each oscillatory mode, and the ground state energy is simply

$$e_0 = \frac{\hbar\omega}{2}$$

Casimir's Derivation

Casimir derived an expression for the "force from nothing" in this way:

Consider a cubic cavity bound by perfect conductors, with side length L. Within that cavity is placed a conducting LxL plate a distance a away from the xy plane. The resonant modes of the cavity are bound by

$$0 \le x \le L,\ 0 \le y \le L,\ 0 \le z \le a$$

yielding wave numbers

$$k_x = \frac{\pi}{L}n_x,\quad k_y = \frac{\pi}{L}n_y,\quad k_z = \frac{\pi}{a}n_z \quad \text{and thus } \mathrm{k} = \sqrt{k_x^2 + k_y^2 + k_z^2}$$

summing over the ground state energy of each mode $\left(\frac{\hbar\omega}{2} \text{ for each frequency}\right)$,

the vacuum energy in the cavity between the square plate and the wall of the xy plane is

$$\frac{1}{2}\sum \hbar\omega = \frac{1}{2}\sum \hbar ck = \hbar c\frac{L^2}{\pi^2}\iint\left[\frac{1}{2}\sqrt{k_x^2 + k_y^2} + \sum_{n=1}^{\infty}\sqrt{n^2\frac{\pi^2}{a^2} + k_x^2 + k_y^2}\right]dk_x dk_y$$

where the sum over *kx* and *ky* has been replaced with an integral, as *L* is large enough that they can be considered continuous. Also, a factor of two has been introduced to account for the two polarizations of all frequencies except where one of the *k* values is zero, where there is only one standing wave. This has been ignored for *kx* and *ky* because they are continuous.

Casimir wanted to consider the difference between the energy represented by this expression, where the plate is very close to the *xy* wall, and the case where it is far away. For large *a* (where *kz* can also be treated as continuous):

$$\frac{1}{2}\sum \hbar\omega = \hbar c \frac{L^2}{\pi^2}\frac{a}{\pi}\iiint \sqrt{k_z^2 + k_x^2 + k_y^2}\, dk_z dk_x dk_y$$

Each of these expressions is infinite but their difference can be evaluated. Doing so requires a term that goes to zero as the frequencies go to infinity, which is reasonable because very energetic photons will be unimpeded by the conducting plates.

After these tricks, he wound up with a force $F = \hbar c \frac{\pi^2}{240}\frac{1}{a^4}$. It was especially remarkable that the force depended only on fundamental constants.

Further Developments and Experimental Confirmation

In 1958, Sparnaay et al published a paper discussing measurements of attractive forces between flat plates like those in Casimir's theory. Due to difficulties aligning the plates and measuring the force between them, all Sparnaay could say was that his results "did not contradict Casimir's theoretical prediction" [3]. In 1961, a paper entitled *General Theory of Van der Waals' Forces*, Lifshits *et al* made a more general analysis of the force, including considerations of temperature and finite conductivity. Difficulties arising from surface roughness and geometric factors were also considered [4]. A study by Boyer revealed that the case of a conducting spherical shell that is divided into two hemispheres yields the unintuitive result of a repulsive Casimir force [5]. After a substantial gap in experimental observations, an attempt to measure the force was made in 1997 at the University of Washington, where Lamoreaux measured the Casimir force between a plane and a sphere, 4cm in diameter, at a distance of 0.6μm to 6μm apart. This experiment agreed with theory to within 5% [6]. Another attempt to measure the effect, this time using atomic force microscopy, was made in 1998. It was again between a plane and a sphere—the sphere was attached to the AFM cantilever and had a radius of only 100μm—at a distance of only 60nm to 500nm. This experiment differed from theoretical predictions by less than 2% [7].

Significance in MEMS

As the Casimir force is dominant in the micrometer range, it naturally has significant implications for microelectromechanical systems (MEMS) and nano-scale fabrication. Indeed, some of the measurements of the Casimir force mentioned above used MEMS technology. MEMS devices often take advantage of structures such as micro-scale membrane strips because they are easy to manufacture and very useful, but in many cases the geometry and flexibility of such elements leaves them prone to the negative influence of "stiction", a tendency of small MEMS elements to adhere to neighboring surfaces due to Casimir or Van der Waals forces [8]. Careful analysis of the Casimir force is necessary to design devices that avoid such problems.

Besides the extreme case of stiction, there are instances where the Casimir effect can have a significant influence on MEMS operation in general. There are cases where the Casimir force must be considered with non-trivial materials and exotic geometries in order to optimize all kinds of MEMS devices [9]. This doesn't mean simply avoiding the problem of stiction, but using precise models of the Casimir force to actually take advantage of the effect in various ways.

Cosmological Implications

The Casimir effect has been pointed to as a demonstration of the "realness" of vacuum state energy, which has significant implications in cosmology. While Einstein was persuaded to include a "cosmological constant" term in his theory of relativity to ensure that the universe remained static, the discovery of Doppler shifts in the spectra of stars indicated that the universe is indeed expanding, confirming Einstein's constant-free formulation. However, more recent observations have shown that the expansion of the universe is in fact accelerating, prompting the reemergence of the cosmological constant. Since energy gravitates, the existence of a large amount of vacuum state energy such as that involved in the Casimir effect has been proposed as a candidate to explain the new cosmological constant [10]. The problem is that an integrated count of all the vacuum energy in the universe certainly diverges, whereas the observed small, outward acceleration indicates that the cosmological constant should be a small, positive number [11].

Is Vacuum Energy Real?

These cosmological concerns relate to the more basic question of whether the Casimir effect really proves that this energy in the vacuum is "real". Similar questions over the "realness" of a mathematical construct have arisen before in physics. A good example is the effect of the vector potential in a region with zero field in the Aharonov Bohm effect [12]. Where zero point energy is concerned, a recent (2005) paper by MIT physicist Brian Jaffe denies the special significance of the effect. He begins by highlighting several cases where prominent cosmologists have invoked the effect to justify the "realness" of vacuum energy [13]:

"Perhaps surprisingly, it was along time before particle physicists began se-
riously to worry about [quantum zero point fluctuation contributions to [the cosmological constant] despite the demonstration in the Casimir effect of the reality of zero-point energies." [14-Weinberg]

" ... And the vacuum fluctuations themselves are very real, as evidenced by the Casimir effect." [15-Carrol]

"The existence of zero-point vacuum fluctuations has been spectacularly demonstrated by the Casimir effect." [16-Sahni]

Jaffe maintains that invoking zero-point energy to explain the Casimir effect, while elegant, is not the only way to explain the force, nor is it revealing of a deeper significance such energy. He cites an explanation given by Schwinger in 1974, which rederived the Casimir effect using source theory and without invoking zero-point energy.

This model also contained the appropriate temperature and conductivity dependence absent from Casimir's early paper (though not from subsequent work by Lifshitz and others). Jaffe further points out that in this formulation, the force is a function of the fine structure constant, and that the physically impossible perfect-conductor case is analogous to the limit where the fine structure constant goes to infinity. Also, Jaffe claims, like all dynamic effects of QED, the Casimir effect disappears in the limit where the fine structure constant approaches zero. Clearly, Jaffe concludes that the Casimir effect alone is not sufficient evidence that there is anything physical about vacuum energy, and that it is still possible such energy is just a meaningless infinity that can safely be subtracted out normalization [13].

References

1. H.B.G. Casimir, D. Polder, Phys. Rev. **73** (1948) 360.
2. H. B. G. Casimir, Kon. Ned. Akad. Wetensch. Proc. **51** (1948) 793.
3. M.J. Sparnaay, Physica **24** (1958) 751.
4. I.E. Dzyaloshinskii, E.M. Lifshitz, L.P. Pitaevskii, Sov. Phys.—Usp. (USA) **4** (1961) 153.
5. T.H. Boyer, Phys. Rev. **174** (1968) 1764.
6. S.K. Lamoreaux, Phys. Rev. Lett. **78** (1997) 5.
7. U. Mohideen, A. Roy, Phys Rev. Lett. **81** (1998) 21.
8. F. Serry, D. Walliser, G.J. Maclay, J. Appl. Phys. **84** (1998) 2501.
9. H. B. Chan et al, Phys. Rev. Lett. **87**, 211801 (2001).
10. Zeldovich, Y., JETP Lett. **6**, 316 (1967) [Pisma Zh. Eksp. Teor. Fiz. **6**, 883 (1967)].
11. Tegmark, M. *et al.* [SDSS Collaboration], Phys. Rev. D **69**, 103501 (2004).
12. Aharonov, Y.; Bohm D. (1959). "Significance of electromagnetic potentials in quantum theory". *Phys. Rev.* **115**: 485–491.
13. R. L. Jaffe, Phys.Rev. **D72** (2005) 021301 [arXiv:hep-th/0503158v1].
14. S. Weinberg, Rev. Mod. Phys. **61**, 1 (1989).
15. S. M. Carroll, Living Rev. Rel. **4**, 1 (2001).
16. V. Sahni and A. A. Starobinsky, Int. J. Mod. Phys. D **9**, 373 (2000).