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Levi-Civita symbol

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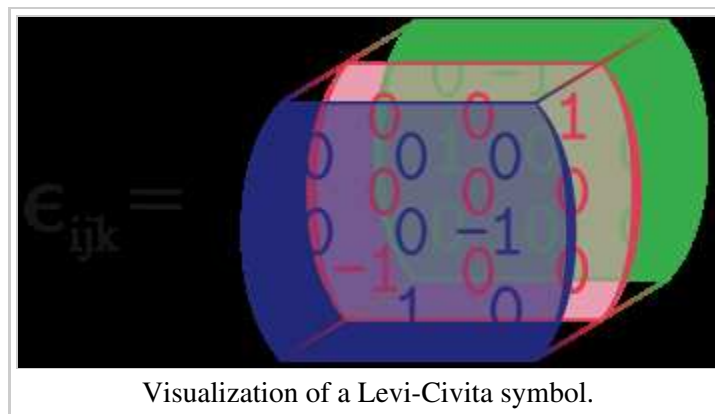
The **Levi-Civita symbol**, also called the **permutation symbol** or **antisymmetric symbol**, is a mathematical symbol used in particular in tensor calculus. It is named after the Italian mathematician and physicist Tullio Levi-Civita.

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Definition

In three dimensions, the Levi-Civita symbol is defined as follows:



$$\varepsilon_{ijk} = \begin{cases} +1 & \text{if } (i, j, k) \text{ is } (1, 2, 3), (2, 3, 1) \text{ or } (3, 1, 2), \\ -1 & \text{if } (i, j, k) \text{ is } (3, 2, 1), (1, 3, 2) \text{ or } (2, 1, 3), \\ 0 & \text{otherwise: } i = j \text{ or } j = k \text{ or } k = i, \end{cases}$$

i.e. it is 1 if (i, j, k) is an even permutation of $(1,2,3)$, -1 if it is an odd permutation, and 0 if any index is repeated.

For example, in linear algebra, the determinant of a 3×3 matrix A can be written

$$\det A = \sum_{i,j,k=1}^3 \varepsilon_{ijk} a_{1i} a_{2j} a_{3k}$$

(and similarly for a square matrix of general size, see below)

and the cross product of two vectors can be written as a determinant:

$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = \sum_{i,j,k=1}^3 \varepsilon_{ijk} \mathbf{e}_i a_j b_k$$

or more simply:

$$\mathbf{a} \times \mathbf{b} = \mathbf{c}, \quad c_i = \sum_{j,k=1}^3 \varepsilon_{ijk} a_j b_k.$$

According to the Einstein notation, the summation symbol may be omitted.

The tensor whose components are given by the Levi-Civita symbol (a tensor of covariant rank n) is sometimes called the **permutation tensor**. It is actually a pseudotensor because under an orthogonal transformation of jacobian determinant -1 (i.e., a rotation composed with a reflection), it gets a -1 . Because the Levi-Civita symbol is a pseudotensor, the result of taking a cross product is a pseudovector, not a vector.

Relation to Kronecker delta

The Levi-Civita symbol is related to the Kronecker delta. In three dimensions, the relationship is given by the following equations:

$$\begin{aligned} \varepsilon_{ijk} \varepsilon_{lmn} &= \det \begin{vmatrix} \delta_{il} & \delta_{im} & \delta_{in} \\ \delta_{jl} & \delta_{jm} & \delta_{jn} \\ \delta_{kl} & \delta_{km} & \delta_{kn} \end{vmatrix} \\ &= \delta_{il} (\delta_{jm} \delta_{kn} - \delta_{jn} \delta_{km}) - \delta_{im} (\delta_{jl} \delta_{kn} - \delta_{jn} \delta_{kl}) + \delta_{in} (\delta_{jl} \delta_{km} - \delta_{jm} \delta_{kl}) \\ \sum_{i=1}^3 \varepsilon_{ijk} \varepsilon_{imn} &= \delta_{jm} \delta_{kn} - \delta_{jn} \delta_{km} \quad (\text{"contracted epsilon identity"}) \\ \sum_{i,j=1}^3 \varepsilon_{ijk} \varepsilon_{ijn} &= 2\delta_{kn} \end{aligned}$$

Generalization to n dimensions

The Levi-Civita symbol can be generalized to higher dimensions:

$$\varepsilon_{ijkl\dots} = \begin{cases} +1 & \text{if } (i, j, k, \ell, \dots) \text{ is an even permutation of } (1, 2, 3, 4, \dots) \\ -1 & \text{if } (i, j, k, \ell, \dots) \text{ is an odd permutation of } (1, 2, 3, 4, \dots) \\ 0 & \text{if any two labels are the same} \end{cases}$$

Thus, it is the sign of the permutation in the case of a permutation, and zero otherwise.

Furthermore, it can be shown that

$$\sum_{i,j,k,\dots=1}^n \varepsilon_{ijk\dots} \varepsilon_{ijk\dots} = n!$$

is always fulfilled in n dimensions. In index-free tensor notation, the Levi-Civita symbol is replaced by the concept of the Hodge dual.

In general n dimensions one can write the product of two Levi-Civita symbols as:

$$\varepsilon_{ijk\dots} \varepsilon_{mnl\dots} = \det \begin{vmatrix} \delta_{im} & \delta_{in} & \delta_{il} & \dots \\ \delta_{jm} & \delta_{jn} & \delta_{jl} & \dots \\ \delta_{km} & \delta_{kn} & \delta_{kl} & \dots \\ \vdots & \vdots & \vdots & \ddots \end{vmatrix}.$$

Now we can contract m indexes, this will add a $m!$ factor to the determinant and we need to omit the relevant Kronecker delta.

Properties

(superscripts should be considered equivalent with subscripts)

1. When $n = 2$, we have for all i,j,m,n in $\{1,2\}$,

$$\varepsilon_{ij} \varepsilon^{mn} = \delta_i^m \delta_j^n - \delta_i^n \delta_j^m, \quad (1)$$

$$\varepsilon_{ij} \varepsilon^{in} = \delta_j^n, \quad (2)$$

$$\varepsilon_{ij} \varepsilon^{ij} = 2. \quad (3)$$

2. When $n = 3$, we have for all i,j,k,m,n in $\{1,2,3\}$,

$$\varepsilon_{jmn} \varepsilon^{imn} = 2\delta_j^i, \quad (4)$$

$$\varepsilon_{ijk} \varepsilon^{ijk} = 6. \quad (5)$$

Proofs

For equation 1, both sides are antisymmetric with respect of ij and mn . We therefore only need to consider the case $i \neq j$ and $m \neq n$. By substitution, we see that the equation holds for $\varepsilon_{12} \varepsilon^{12}$, i.e., for $i = m = 1$ and $j = n = 2$. (Both sides are then one). Since the equation is antisymmetric in ij and mn , any set of values for these can be reduced to the above case (which holds). The equation thus holds for all values of ij and mn . Using equation 1, we have for equation 2 $\varepsilon_{ij} \varepsilon^{in} = \delta_i^i \delta_j^n - \delta_i^n \delta_j^i$

$$= 2\delta_j^n - \delta_j^n$$

$$= \delta_j^n.$$

Here we used the Einstein summation convention with i going from 1 to 2. Equation 3 follows similarly from equation 2. To establish equation 4, let us first observe that both sides vanish when $i \neq j$. Indeed, if $i \neq j$, then one can not choose m and n such that both permutation symbols on the left are nonzero. Then, with $i = j$ fixed, there are only two ways to choose m and n from the remaining two indices. For any such indices, we have $\varepsilon_{jmn}\varepsilon^{imn} = (\varepsilon^{imn})^2 = 1$ (no summation), and the result follows. The last property follows since $3! = 6$ and for any distinct indices i, j, k in $\{1, 2, 3\}$, we have $\varepsilon_{ijk}\varepsilon^{ijk} = 1$ (no summation). \square

Examples

1. The determinant of an $n \times n$ matrix $A = (a_{ij})$ can be written as

$$\det A = \varepsilon_{i_1 \dots i_n} a_{1i_1} \cdots a_{ni_n},$$

where each i_l should be summed over $1, \dots, n$.

Equivalently, it may be written as

$$\det A = \frac{1}{n!} \varepsilon_{i_1 \dots i_n} \varepsilon_{j_1 \dots j_n} a_{i_1 j_1} \cdots a_{i_n j_n},$$

where now each i_l and each j_l should be summed over $1, \dots, n$.

2. If $A = (A^1, A^2, A^3)$ and $B = (B^1, B^2, B^3)$ are vectors in R^3 (represented in some right hand oriented orthonormal basis), then the i th component of their cross product equals

$$(A \times B)^i = \varepsilon^{ijk} A^j B^k.$$

For instance, the first component of $A \times B$ is $A^2 B^3 - A^3 B^2$. From the above expression for the cross product, it is clear that $A \times B = -B \times A$. Further, if $C = (C^1, C^2, C^3)$ is a vector like A and B , then the triple scalar product equals

$$A \cdot (B \times C) = \varepsilon^{ijk} A^i B^j C^k.$$

From this expression, it can be seen that the triple scalar product is antisymmetric when exchanging any adjacent arguments. For example, $A \cdot (B \times C) = -B \cdot (A \times C)$.

3. Suppose $F = (F^1, F^2, F^3)$ is a vector field defined on some open set of R^3 with Cartesian coordinates $x = (x^1, x^2, x^3)$. Then the i th component of the curl of F equals

$$(\nabla \times F)^i(x) = \varepsilon^{ijk} \frac{\partial}{\partial x^j} F^k(x).$$

Notation

A shorthand notation for anti-symmetrization is denoted by a pair of square brackets. For example, for an $n \times n$ matrix, M ,

$$M_{[ab]} = \frac{1}{2}\varepsilon_{ab}\varepsilon^{cd}M_{dc} = \frac{1}{2}(M_{ab} - M_{ba})$$

and for a rank 3 tensor T,

$$T_{[abc]} = \frac{1}{3!}(T_{abc} - T_{acb} + T_{bca} - T_{bac} + T_{cab} - T_{cba})$$

References

- Charles W. Misner, Kip S. Thorne, John Archibald Wheeler, *Gravitation*, (1970) W.H. Freeman, New York; ISBN 0-7167-0344-0. (*See section 3.5 for a review of tensors in general relativity*).

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