SOLUTION TO PROBLEM SET 8; PHYSICS 321

Problem 4.15

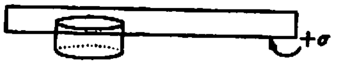
(a) $\rho_b = -\nabla\cdot\mathbf{P} = -\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{k}{r}\right) = -\frac{k}{r^2}$; $\quad \sigma_b = \mathbf{P}\cdot\hat{\mathbf{n}} = \left\{\begin{array}{ll} +\mathbf{P}\cdot\hat{\mathbf{r}} = k/b & (\text{at } r = b), \\ -\mathbf{P}\cdot\hat{\mathbf{r}} = -k/a & (\text{at } r = a). \end{array}\right\}$

Gauss's law $\Rightarrow \mathbf{E} = \frac{1}{4\pi\epsilon_0}\frac{Q_{\text{enc}}}{r^2}\hat{\mathbf{r}}$. For $r < a$, $Q_{\text{enc}} = 0$, so $\boxed{\mathbf{E} = 0.}$ For $r > b$, $Q_{\text{enc}} = 0$ (Prob. 4.14), so $\boxed{\mathbf{E} = 0.}$ For $a < r < b$, $Q_{\text{enc}} = \left(\frac{-k}{a}\right)(4\pi a^2) + \int_a^r \left(\frac{-k}{\bar{r}^2}\right) 4\pi\bar{r}^2 d\bar{r} = -4\pi k a - 4\pi k(r-a) = -4\pi k r$; so $\boxed{\mathbf{E} = -(k/\epsilon_0 r)\,\hat{\mathbf{r}}.}$

(b) $\oint \mathbf{D}\cdot d\mathbf{a} = Q_{f_{\text{enc}}} = 0 \Rightarrow \mathbf{D} = 0$ everywhere. $\mathbf{D} = \epsilon_0\mathbf{E} + \mathbf{P} = 0 \Rightarrow \mathbf{E} = (-1/\epsilon_0)\mathbf{P}$, so $\boxed{\mathbf{E} = 0 \text{ (for } r < a \text{ and } r > b);}$ $\boxed{\mathbf{E} = -(k/\epsilon_0 r)\,\hat{\mathbf{r}} \text{ (for } a < r < b).}$

Problem 4.18

(a) Apply $\int \mathbf{D}\cdot d\mathbf{a} = Q_{f_{\text{enc}}}$ to the gaussian surface shown. $DA = \sigma A \Rightarrow \boxed{D = \sigma.}$ (*Note:* $\mathbf{D} = 0$ inside the metal plate.) This is true in both slabs; $\mathbf{D}$ points *down*.



(b) $\mathbf{D} = \epsilon\mathbf{E} \Rightarrow E = \sigma/\epsilon_1$ in slab 1, $E = \sigma/\epsilon_2$ in slab 2. But $\epsilon = \epsilon_0\epsilon_r$, so $\epsilon_1 = 2\epsilon_0$; $\epsilon_2 = \frac{3}{2}\epsilon_0$. $\boxed{E_1 = \sigma/2\epsilon_0,}$ $\boxed{E_2 = 2\sigma/3\epsilon_0.}$

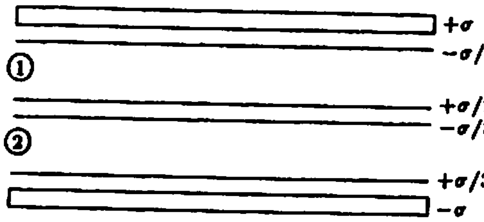
(c) $\mathbf{P} = \epsilon_0\chi_e\mathbf{E}$, so $P = \epsilon_0\chi_e d/(\epsilon_0\epsilon_r) = (\chi_e/\epsilon_r)\sigma$; $\chi_e = \epsilon_r - 1 \Rightarrow P = (1 - \epsilon_r^{-1})\sigma$. $\boxed{P_1 = \sigma/2,}$ $\boxed{P_2 = \sigma/3.}$

(d) $V = E_1 a + E_2 a = (\sigma a/6\epsilon_0)(3+4) = \boxed{7\sigma a/6\epsilon_0.}$

(e) $\rho_b = 0$; $\boxed{\begin{array}{l} \sigma_b = +P_1 \text{ at bottom of slab (1)} = \sigma/2, \\ \sigma_b = -P_1 \text{ at top of slab (1)} = -\sigma/2; \end{array}}$ $\boxed{\begin{array}{l} \sigma_b = +P_2 \text{ at bottom of slab (2)} = \sigma/3, \\ \sigma_b = -P_2 \text{ at top of slab (2)} = -\sigma/3. \end{array}}$

(f) In slab 1: $\left\{\begin{array}{l} \text{total surface charge } \textit{above}: \sigma - (\sigma/2) = \sigma/2, \\ \text{total surface charge } \textit{below}: (\sigma/2) - (\sigma/3) + (\sigma/3) - \sigma = -\sigma/2, \end{array}\right\} \Longrightarrow E_1 = \frac{\sigma}{2\epsilon_0}. \checkmark$

In slab 2: $\left\{\begin{array}{l} \text{total surface charge } \textit{above}: \sigma - (\sigma/2) + (\sigma/2) - (\sigma/3) = 2\sigma/3, \\ \text{total surface charge } \textit{below}: (\sigma/3) - \sigma = -2\sigma/3, \end{array}\right\} \Longrightarrow E_2 = \frac{2\sigma}{3\epsilon_0}. \checkmark$



3. a) In this orientation, the slabs are in series so for two such slabs

$$\begin{aligned}\frac{1}{C} &= \frac{1}{C_1}+\frac{1}{C_2},\\ &= \frac{a}{L_xL_y\epsilon_0}\left(\frac{1}{\kappa_1}+\frac{1}{\kappa_2}\right),\\ &= \frac{a}{L_xL_y\epsilon_0}\frac{\kappa_1+\kappa_2}{\kappa_1\kappa_2}.\end{aligned}$$

There are $d/2a$ such capictors in series so

$$C_{series} = \frac{2A\epsilon_0}{d}\frac{\kappa_1\kappa_2}{\kappa_1+\kappa_2} = \frac{A\epsilon_0}{d}\frac{\kappa_1\kappa_2}{(\kappa_1+\kappa_2)/2},$$

where $A = L_xL_y$.

b) With two plates in parallel and the normal to the slabs parallel to the x axis

$$C = \frac{\epsilon_0 L_y a}{d}(\kappa_1+\kappa_2).$$

There are $L_x/2a$ such capacitors in parallel so

$$C_{parallel} = \frac{L_x}{2a}\frac{\epsilon_0 L_y a}{d}(\kappa_1+\kappa_2) = \frac{A\epsilon_0}{d}\frac{(\kappa_1+\kappa_2)}{2}.$$

To see that $C_{parallel} > C_{series}$

$$\begin{aligned}(\kappa_1-\kappa_2)^2 &\geq 0,\\ \kappa_1^2 - 2\kappa_1\kappa_2 + \kappa_2^2 &\geq 0,\\ \kappa_1^2 + 2\kappa_1\kappa_2 + \kappa_2^2 &\geq 4\kappa_1\kappa_2,\\ (\kappa_1+\kappa_2)^2 &\geq 4\kappa_1\kappa_2 \qquad \text{so that}\\ \frac{\kappa_1+\kappa_2}{2} &\geq \frac{\kappa_1\kappa_2}{(\kappa_1+\kappa_2)/2},\\ C_{parallel} &\geq C_{series}\end{aligned}$$

4.a) Let the dielectric of dielectric constant κ fill the plates from x to L. The system can be treated as two capacitors in parallel. One has area Ax/L and is filled with material of dielectric constant unity. Its capacitance is

$$C_1 = \epsilon_0 Ax/Ld = C(L)x/L.$$

The other has plates of area $A(1 - x/L)$ and is filled with material of dielectric constant κ, so that its capacitance is

$$C_2 = C(L)\kappa\left(1 - \frac{x}{L}\right).$$

Therefore

$$C_{equiv} = C_1 + C_2 = C(L)\left[\frac{x}{L} + \kappa\left(1 - \frac{x}{L}\right)\right] = C(L)\left(1 + (\kappa - 1)\left[1 - \frac{x}{L}\right]\right)$$

b)

$$\begin{aligned} F_Q(x) &= \frac{Q_0^2}{C^2(x)}\frac{dC(x)}{dx}, \\ &= -\frac{Q_0^2}{2C^2(L)}\frac{(\kappa - 1)/L}{[1 + (\kappa - 1)(1 - x/L)]^2}, \\ &= -\frac{Q_0^2}{2C^2(L)}\frac{\kappa - 1}{[1 + (\kappa - 1)(1 - x/L)]^2}. \qquad \text{In contrast} \\ F_V(x) &= \frac{Q_0^2}{2C^2(L)}\frac{dC(x)}{dx}, \\ &= -\frac{Q_0^2}{LC^2(L)}\frac{\kappa - 1}{2}, \qquad \text{a constant.} \end{aligned}$$