SOLUTION TO PROBLEM SET 7

1. a) From Gauss' Law we know that

$$\sigma(\theta) = \epsilon_0[E(R_+,\theta) - E(R_-,\theta)]$$

and we can calculate the electric fields from the potential. Therefore

$$\begin{aligned}\sigma(\theta) &= \epsilon_0 \left(\sum_l (l+1)\frac{B_l}{R^{l+2}} + l A_l R^{l-1} \right) P_l(\cos\theta), \\ &= \epsilon_0 \sum_l (2l+1) A_l R^{l-1} P_l(\cos\theta).\end{aligned}$$

Multiply both sides of this equation by $P_{l'}(\cos\theta)\sin\theta d\theta$ and integrate from $\theta = 0$ to $\theta = \pi$;

$$\begin{aligned}\int \sigma(\theta) P_{l'} \sin\theta \; d\theta &= \epsilon_0 \sum_l (2l+1) A_l R^{l-1} \int_0^\pi P_l(\cos\theta) P_{l'}(\cos\theta) \sin\theta \; d\theta, \\ &= 2\epsilon_0 A_{l'} R^{l'-1} \qquad \text{so that} \\ A_{l'} &= \frac{1}{2\epsilon_0 R^{l'-1}} \int_0^\pi \sigma(\theta) P_{l'}(\cos\theta) \sin\theta \; d\theta.\end{aligned}$$

b) For $\sigma(\theta) = Q/4\pi R^2 = QP_0(\cos\theta)/4\pi R^2$, we find from

$$\begin{aligned}\int_0^\pi d\theta P_l(\cos\theta) P_{l'}(\cos\theta) \sin\theta \; d\theta &= \frac{2}{2l+1}\delta_{l,l'}, \qquad \text{that} \\ A_l &= \frac{Q}{4\pi\epsilon_0 R}\delta_{l,0} \qquad \text{so that} \\ V(r,\theta) &= \frac{Q}{4\pi\epsilon_0 R} \qquad r \le R, \\ &= \frac{Q}{4\pi\epsilon_0 r} \qquad r \ge R \qquad \text{as it should.}\end{aligned}$$

c) For $\sigma(\theta) = 3\epsilon_0 V_{MAX} P_1(\cos\theta)/R$, we proceed in the same way

$$\begin{aligned}A_l &= \frac{3V_{MAX}}{2R^l} \int_0^\pi P_1(\cos\theta) P_l(\cos\theta) \sin\theta \; d\theta, \\ &= \frac{V_{MAX}}{R}\delta_{1,l} \qquad \text{so that}\end{aligned}$$

$$\begin{aligned} V(r,\theta) &= V_{MAX}\frac{r}{R}\cos\theta, \qquad r \le R, \\ &= V_{MAX}\left(\frac{R}{r}\right)^2\cos\theta \qquad r \ge R. \end{aligned}$$

2. On the surface of the sphere

$$\begin{aligned} V(r,\theta) &= \sum_l A_l R^l P_l(\cos\theta) \qquad \text{so that} \\ A_l &= \frac{2l+1}{2R^l}\int_0^\pi V(\theta)P_l(\cos\theta)\sin\theta\ d\theta, \\ &= \frac{2l+1}{2R^l}\int_0^\pi V_{MAX}\cos^2\theta P_l(\cos\theta)\sin\theta\ d\theta, \\ &= \frac{2l+1}{2R^l}\int_0^\pi V_{MAX}\left(\frac{2P_2(\cos\theta)+P_0(\cos\theta)}{3}\right)P_l(\cos\theta)\sin\theta\ d\theta, \\ &= \frac{2l+1}{2R^l}\left(\frac{2}{3}\frac{2}{5}\delta_{l,2}+\frac{1}{3}2\delta_{l,0}\right), \text{from which} \\ A_0 &= \frac{V_{MAX}}{3}, \\ A_2 &= \frac{2V_{MAX}}{3R^2}, \\ A_l &= 0 \qquad l \ne 0,2. \qquad \text{Thus} \\ V(r,\theta) &= V_{MAX}\left(\frac{1}{3}+\frac{2}{3}\left(\frac{r}{R}\right)^2 P_2(\cos\theta)\right) \qquad r \le R, \\ &= V_{MAX}\left(\frac{1}{3}+\frac{2}{3}\left(\frac{R}{r}\right)^3 P_2(\cos\theta)\right) \qquad r \ge R. \end{aligned}$$

3. Griffiths 3.22.

$$\begin{aligned} \sigma(\theta) &= \sigma_0 \qquad \pi/2 < \theta \le 0, \\ &= -\sigma) \qquad \pi/2 < \theta \le \pi \end{aligned}$$

We know that

$$A - l = \frac{1}{2\epsilon_0 R^{l-1}}\int_0^\pi \sigma(\theta)P_l(\cos\theta)\sin\theta\ d\theta$$

Because $\sigma(\theta)$ is odd about $\theta = \pi/2$, only A_l with l odd will be non-zero. Inside the sphere all B_l vanish. Thus we can write

$$\begin{aligned} A_l &= \frac{\sigma_0}{\epsilon_0 R^{l-1}} \int_0^{\pi/2} P_l(\cos\theta) \sin\theta \; d\theta, \\ &= \frac{\sigma_0}{\epsilon_0 R^{l-1}} \int_0^1 P_l(x) dx, \end{aligned}$$

With

$$\begin{aligned} P_i(x) &= x, \\ P_3(x) &= (5x^3 - 3x)/2, \\ P_5(x) &= (63x^5 - 7 - x^3 + 15x)/8, \end{aligned}$$

I get, on carrying out the integrals

$$\begin{aligned} A_1 &= \sigma_0/2\epsilon_0, \\ A_3 &= -\sigma_0/8\epsilon_0 R^2, \\ A_5 &= 1\sigma_0/16\epsilon_0 R^4, \qquad \text{so that} \\ V(r,\theta) &= \frac{\sigma_0 R}{\epsilon_0} \left\{ \frac{1}{2} \left(\frac{r}{R}\right) P_1(\cos\theta) - \frac{1}{8} \left(\frac{r}{R}\right)^3 P_3(\cos\theta) + \frac{1}{16} \left(\frac{r}{R}\right)^5 P_5(\cos\theta) \right\}, \qquad r \leq R \end{aligned}$$

Similarly, outside the sphere all the A_l vanish and the non-zero B_l are related to the non-zero A_l above by $B_l = A_l R^{2l+1}$. Thus I obtain

$$V(r,\theta) = \frac{\sigma_0 R}{\epsilon_0} \left\{ \frac{1}{2} \left(\frac{R}{r}\right)^2 P_1(\cos\theta) - \frac{1}{8} \left(\frac{R}{r}\right)^4 P_3(\cos\theta) + \frac{1}{16} \left(\frac{R}{r}\right)^6 P_5(\cos\theta) \right\}, \qquad r \geq R$$