PROBLEM SET 6: DUE MAY 20

1)† Solve the same system in Prob. set 5, problem 1, by the method of separation of variables. You will obtain an infinite series of course. Evaluate your solution on a 12 by 12 grid with the first and last rows and columns fixed by the boundary conditions as in Problem Set 5. You will have to choose a maximum integer, n_{max}, at which you truncate the sum. Choose it by the same criteria i.e. the root-means-square change in potential on increasing N_{max} by unity is less than 0.001. Print out your results.

2)† Consider a non-conducting cylindrical pipe of radius R with axis in the z direction. The charge density on the pipe is even in the angle θ

$$\sigma(\theta) = \sigma(-\theta).$$

Clearly the potential $V(s,\theta)$ is therefore an even function of θ.

a) Write down, in the form of an infinite series, the most general solution of Laplace's equation subject to the boundary conditions that $V(0,\theta)$ is finite, $V(s,\theta)$ goes to zero as s increases without limit, and $V(s,\theta) = V(s,-\theta)$.

b) From Gauss' law we know that

$$\begin{aligned} \sigma(\theta) &= \epsilon_0[E_s(R_+,\theta) - E_s(R_-,\theta)], \\ &= -\left(\frac{\partial V(s,\theta)}{\partial s}\right)_{R_+} + \left(\frac{\partial V(s,\theta)}{\partial s}\right)_{R_-} \end{aligned}$$

Assume that the charge density $\sigma(\theta)$ is known, and solve for the coefficients in the infinite series in terms of $\sigma(\theta)$.

c) Now let the charge density be $\sigma(\theta) = \sigma_0 \cos(2\theta)$, and obtain the solution for the potential both inside and outside of the cylinder.

3).† the same cylinder as in the problem above but now with a charge distribution upon it which is odd in the angle θ. The most general solution for the potential is

$$\begin{aligned} V(s,\theta) &= \sum_{n=1}^{\infty} C_n \left(\frac{s}{R}\right)^n \frac{\sin n\theta}{\sqrt{\pi}} \qquad s \le R, \\ V(s,\theta) &= \sum_{n=1}^{\infty} C_n \left(\frac{R}{s}\right)^n \frac{\sin n\theta}{\sqrt{\pi}} \qquad s \ge R. \end{aligned}$$

a) Calculate the electric field $\mathbf{E}(s,\theta)$ both inside and outside of the cylinder in the form of an infinite series.

b) The energy per unit length of the system can be written

$$U = \frac{\epsilon_0}{2} \int_0^{\infty} ds\ s \int_0^{2\pi} d\theta\ E(s,\theta)^2$$

Obtain an expression for this energy per unit length in the form of an infinite series, and show that the total energy per unit length inside the cylinder is exactly equal to that outside the cylinder. (Wierd!) I found (if I didn't make a mistake) that

$$U = 2\epsilon_0 \sum_{n=1}^{\infty} nC_n^2 \equiv \sum_{n=1}^{\infty} u_n C_n^2$$

where u_n is the energy per unit length of the n'th mode with an amplitude of unity. This shows that the energy can be written as a sum of the energies of each mode weighted by the square of the amplitude of that mode. This is extremely common in Physics. It is the same for the energy in a string, or the energy in an atom (see Quantum Mechanics.)