SOLUTION TO PROBLEM SET 5

1. I wrote a fortran program which took 204 iterations to converge to the desrired tolerance.

120.	240.00	240.00	240.00	240.00	240.00	240.00	240.00	240.00	240.00	240.00	240.
0.	120.00	167.50	190.03	202.68	210.93	217.07	222.17	226.79	231.22	235.61	240.
0.	72.48	119.99	149.94	169.76	183.98	195.18	204.82	213.77	222.50	231.22	240.
0.	49.95	90.03	119.98	142.44	160.06	174.84	188.17	200.97	213.77	226.79	240.
0.	37.30	70.20	97.50	119.97	138.99	155.96	172.07	188.17	204.82	222.17	240.
0.	29.04	55.97	79.88	100.94	119.96	137.96	155.96	174.84	195.18	217.07	240.
0.	22.91	44.78	65.10	83.96	101.96	119.96	138.99	160.06	183.98	210.93	240.
0.	17.81	35.14	51.77	67.86	83.96	100.94	119.97	142.44	169.76	202.68	240.
0.	13.19	26.20	38.98	51.77	65.10	79.88	97.50	119.98	149.94	190.03	240.
0.	8.76	17.48	26.20	35.14	44.78	55.97	70.20	90.03	119.99	167.50	240.
0.	4.38	8.76	13.19	17.81	22.91	29.04	37.30	49.95	72.48	120.00	240.
0.	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	120.

Problem 2.36

(a) $\boxed{\sigma_a = -\frac{q_a}{4\pi a^2};}$ $\boxed{\sigma_b = -\frac{q_b}{4\pi b^2};}$ $\boxed{\sigma_R = \frac{q_a + q_b}{4\pi R^2}.}$

(b) $\boxed{\mathbf{E}_{\text{out}} = \frac{1}{4\pi\epsilon_0}\frac{q_a+q_b}{r^2}\,\hat{\mathbf{r}},}$ **where $\mathbf{r}$ = vector from center of large sphere.**

(c) $\boxed{\mathbf{E}_a = \frac{1}{4\pi\epsilon_0}\frac{q_a}{r_a^2}\,\hat{\mathbf{r}}_a, \quad \mathbf{E}_b = \frac{1}{4\pi\epsilon_0}\frac{q_b}{r_b^2}\,\hat{\mathbf{r}}_b,}$ **where $\mathbf{r}_a$ ($\mathbf{r}_b$) is the vector from center of cavity a (b).**

(d) Zero.

(e) σ_R changes (but not σ_a or σ_b); $\mathbf{E}_{\text{outside}}$ changes (but not $\mathbf{E}_a$ or $\mathbf{E}_b$); force on q_a and q_b still zero.

3. Use Gauss' law to get the electric field in between the two cylinders:

$$\begin{aligned}
E(s)2\pi sL &= \lambda L/\epsilon_0, \\
E(s) &= \frac{\lambda}{2\pi\epsilon_0 s} \qquad R_{inner} \le s \le R_{outer} \text{ from which} \\
V_{in} - V_{out} &= \Delta V = -\int_{R_{outer}}^{R_{inner}} E(s)ds, \\
&= \frac{\lambda}{2\pi\epsilon_0}\ln\left(\frac{R_{outer}}{R_{inner}}\right), \\
&= \frac{\lambda}{2\pi\epsilon_0}\ln\left(1+\frac{d}{R_{inner}}\right). \\
C &= \frac{Q}{\Delta V} = \frac{\lambda L 2\pi\epsilon_0}{\lambda \ln\left(1+\frac{d}{R_{inner}}\right)}.
\end{aligned}$$

For $d/R_{inner} << 1$, $\ln(1+d/R_{inner}) \approx d/r_{inner}$ so that

$$\begin{aligned}
C &\approx \frac{2\pi R_{inner} L\epsilon_0}{d} = \frac{A\epsilon_0}{d}, \\
\frac{C}{A} &= \frac{\epsilon_0}{d} = \frac{8.85\times 10^{-3} F/m}{10^{-9} m} = 0.885\times 10^{-2} F/m^2
\end{aligned}$$

Problem 2.48

(a) $\nabla^2 V = -\frac{\rho}{\epsilon_0}$ (Eq. 2.24), so $\boxed{\dfrac{d^2V}{dx^2} = -\dfrac{1}{\epsilon_0}\rho.}$

(b) $qV = \frac{1}{2}mv^2 \to \boxed{v = \sqrt{\dfrac{2qV}{m}}.}$

(c) $dq = A\rho\, dx$; $\frac{dq}{dt} = a\rho\frac{dx}{dt} = \boxed{A\rho v = I}$ (constant). (*Note:* ρ, hence also I, is *negative.*)

(d) $\frac{d^2V}{dx^2} = -\frac{1}{\epsilon_0}\rho = -\frac{1}{\epsilon_0}\frac{I}{Av} = -\frac{I}{\epsilon_0 A}\sqrt{\frac{m}{2qV}} \Rightarrow \boxed{\dfrac{d^2V}{dx^2} = \beta V^{-1/2}}$, where $\beta = -\frac{I}{\epsilon_0 A}\sqrt{\frac{m}{2q}}$.
(*Note:* I is *negative*, so β is positive; q is *positive.*)

(e) Multiply by $V' = \frac{dV}{dx}$:

$$V'\frac{dV'}{dx} = \beta V^{-1/2}\frac{dV}{dx} \Rightarrow \int V'\, dV' = \beta \int V^{-1/2}\, dV \Rightarrow \frac{1}{2}V'^2 = 2\beta V^{1/2} + \text{constant}.$$

But $V(0) = V'(0) = 0$ (cathode is at potential zero, and field at cathode is zero), so the constant is zero, and

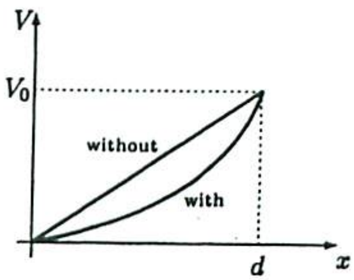
$$V'^2 = 4\beta V^{1/2} \Rightarrow \frac{dV}{dx} = 2\sqrt{\beta}\, V^{1/4} \Rightarrow V^{-1/4}dV = 2\sqrt{\beta}\, dx;$$

$$\int V^{-1/4}\, dV = 2\sqrt{\beta}\int dx \Rightarrow \frac{4}{3}V^{3/4} = 2\sqrt{\beta}\, x + \text{constant}.$$

But $V(0) = 0$, so this constant is also zero.

$$V^{3/4} = \frac{3}{2}\sqrt{\beta}\, x, \text{ so } V(x) = \left(\frac{3}{2}\sqrt{\beta}\right)^{4/3} x^{4/3}, \text{ or } V(x) = \left(\frac{9}{4}\beta\right)^{2/3} x^{4/3} = \left(\frac{81 I^2 m}{32\epsilon_0^2 A^2 q}\right)^{1/3} x^{4/3}.$$

Interms of V_0 (instead of I): $\boxed{V(x) = V_0\left(\dfrac{x}{d}\right)^{4/3}}$ (see graph).
Without space-charge, V would increase linearly: $V(x) = V_0\left(\frac{x}{d}\right)$.



$$\rho = -\epsilon_0\frac{d^2V}{dx^2} = -\epsilon_0 V_0 \frac{1}{d^{4/3}}\frac{4}{3}\cdot\frac{1}{3}x^{-2/3} = \boxed{-\frac{4\epsilon_0 V_0}{9(d^2x)^{2/3}}.}$$

$$v = \sqrt{\frac{2q}{m}}\sqrt{V} = \boxed{\sqrt{2qV_0/m}\left(\frac{x}{d}\right)^{2/3}.}$$

(f) $V(d) = V_0 = \left(\frac{81 I^2 m}{32\epsilon_0^2 A^2 q}\right)^{1/3} d^{4/3} \Rightarrow V_0^3 = \frac{81 m d^4}{32\epsilon_0^2 A^2 q} I^2$; $I^2 = \frac{32\epsilon_0^2 A^2 q}{81 m d^4} V_0^3$;

$I = \frac{4\sqrt{2}\,\epsilon_0 A\sqrt{q}}{9\sqrt{m}\, d^2} V_0^{3/2} = K V_0^{3/2}$, where $\boxed{K = \dfrac{4\epsilon_0 A}{9d^2}\sqrt{\dfrac{2q}{m}}.}$

5.

$$\begin{aligned}
q_{ind} &= \int_0^{\pi} \sigma(\theta) 2\pi a^2 \sin\theta d\theta, \\
&= -\frac{q}{2}\left(1 - \frac{d^2}{a^2}\right)\int_0^{\pi} \frac{\sin\theta d\theta}{[1 - 2(d/a)\cos\theta + (d/a)^2]^{3/2}}, \\
&= \frac{q}{2}\left(1 - \frac{d^2}{a^2}\right)\frac{a}{d}\left[\frac{1}{(1 + 2d/a + d^2/a^2)^{1/2}} - \frac{1}{(1 - 2d/a + d^2/a^2)^{1/2}}\right], \\
&= \frac{q}{2}\left(1 - \frac{d^2}{a^2}\right)\frac{a}{d}\left[\frac{1}{(1 + d/a)} - \frac{1}{(1 - d/a)}\right], \\
&= \frac{q}{2}\left(1 - \frac{d^2}{a^2}\right)\frac{a}{d}\frac{(-2d/a)}{1 - d^2/a^2}, \\
&= -q.
\end{aligned}$$

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In[3]:= f[theta_, r_] := (1 - r * r) / 2 * (1 / (1 - 2 * r * Cos[theta] + r * r))
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In[4]:= Plot[f[theta, .1], {theta, 0, Pi}]
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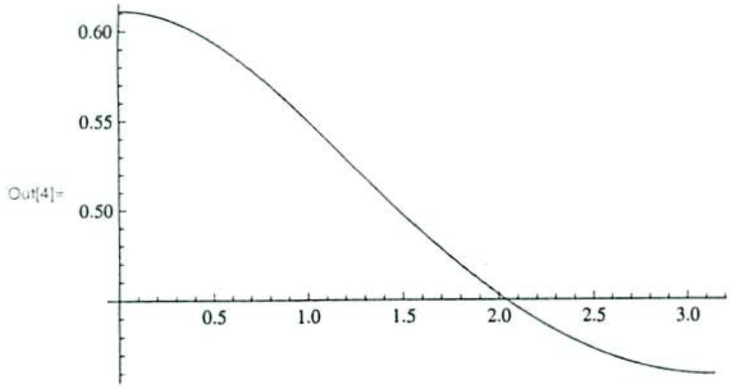


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In[5]:= Plot[f[theta, .5], {theta, 0, Pi}]
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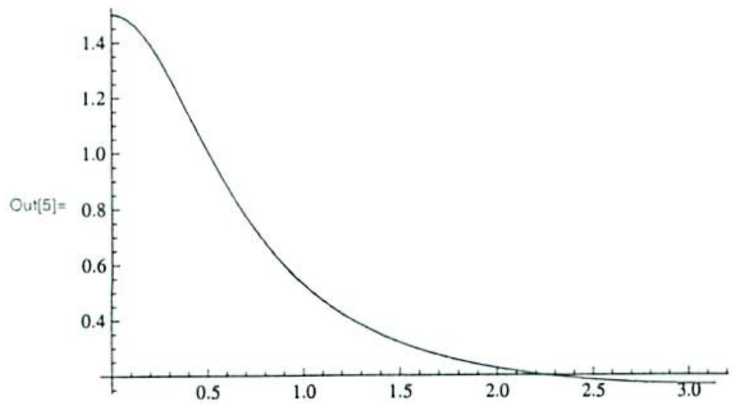


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In[6]:= Plot[f[theta, .9], {theta, 0, Pi}]
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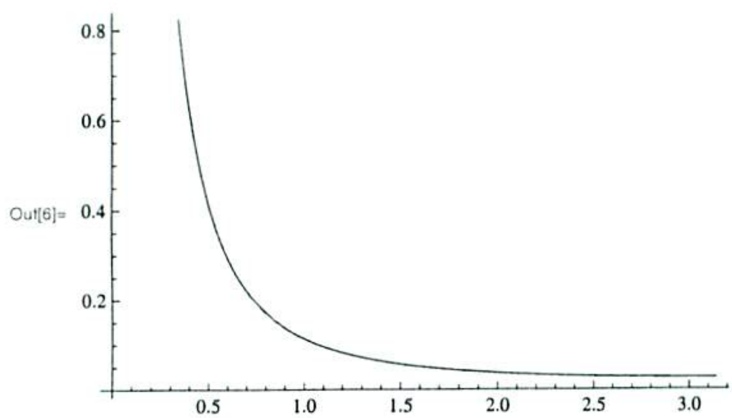

6. If the sphere is electrically neutral, we must add to the previous solution a uniform charge density $\sigma = q/4\pi a^2$ on the outer part of the shell. The total potential is the sum of the previous solution plus the potential produced by this uniform distribution. Thus the total solution is

$$\begin{aligned}
\sigma(\theta) &= \frac{q}{4\pi a^2}\left[1 - \frac{1 - d^2/a^2}{[1 - 2d/a\cos\theta + (d/a)^2]^{3/2}}\right], \\
V(r,\theta) &= V_{old}(r,\theta) + \frac{q}{4\pi\epsilon_0 r} = \frac{q}{4\pi\epsilon_0 r} \qquad r \geq a, \\
&= V_{old}(r,\theta) + \frac{q}{4\pi\epsilon_0 a}, \qquad r \leq a.
\end{aligned}$$