SOLUTION TO PROBLEM SET 4

1. a) The potential that the shell of radius b and charge Q feels due to the shell of radius a is

$$V(r=b) = -\frac{Q}{4\pi\epsilon_0 b}$$

so the potential energy is

$$W_{assemb;e} = Q\frac{(-Q)}{4\pi\epsilon_0 b} = -\frac{Q^2}{4\pi\epsilon_0 b}.$$

Equivalently, one could think of the shell of radius b providing a potential inside of it of

$$V(r=a) = \frac{Q}{4\pi\epsilon_0 b}$$

Thus with the shell of radius a and charge $-Q$ the potential energy is

$$W_{assemble} = (-Q)\frac{Q}{4\pi\epsilon_0 b} = -\frac{Q^2}{4\pi\epsilon_0 b}$$

or one could consider them both, double counting them, and multiplying by 1/2

$$W_{assemble} = \frac{1}{2}\left[Q\frac{(-Q)}{4\pi\epsilon_0 b} + (-Q)\frac{Q}{4\pi\epsilon_0 b}\right] = -\frac{Q^2}{4\pi\epsilon_0 b}$$

b) The energy to create a shell of radius a and charge $(-Q)$ is

$$\begin{aligned} U_a &= \frac{\epsilon_0}{2}\int E^2 d^3r, \\ E(r) &= -\frac{Q}{4\pi\epsilon_0 r^2}, \\ U_a &= \frac{Q^2}{8\pi\epsilon_0 a}. \end{aligned}$$

Similarly the energy to create the shell of radius b and charge Q is

$$U_b = \frac{Q^2}{8\pi\epsilon_0 b},$$

so

$$W_{create} = \frac{Q^2}{8\pi\epsilon_0}\left[\frac{1}{a} + \frac{1}{b}\right]$$

.

c)

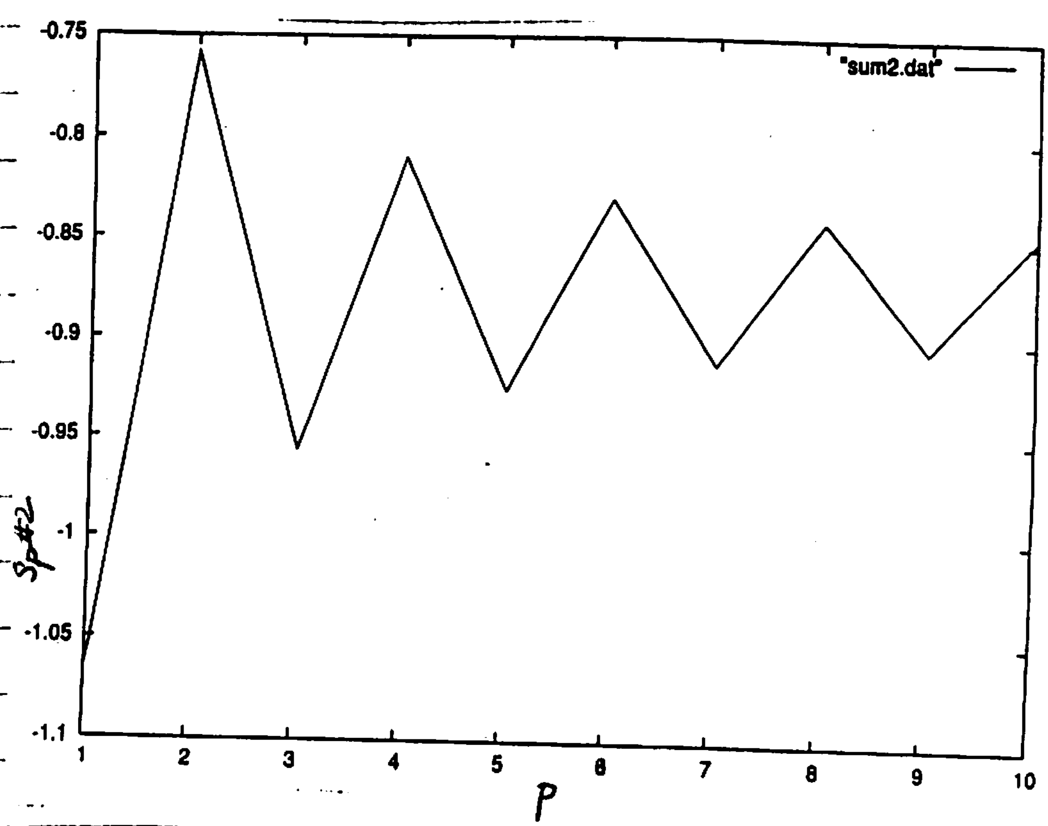
$$W_{tot} = W_{create} + W_{assemble} = \frac{Q^2}{8\pi\epsilon_0}\left[\frac{1}{a} - \frac{1}{b}\right]$$

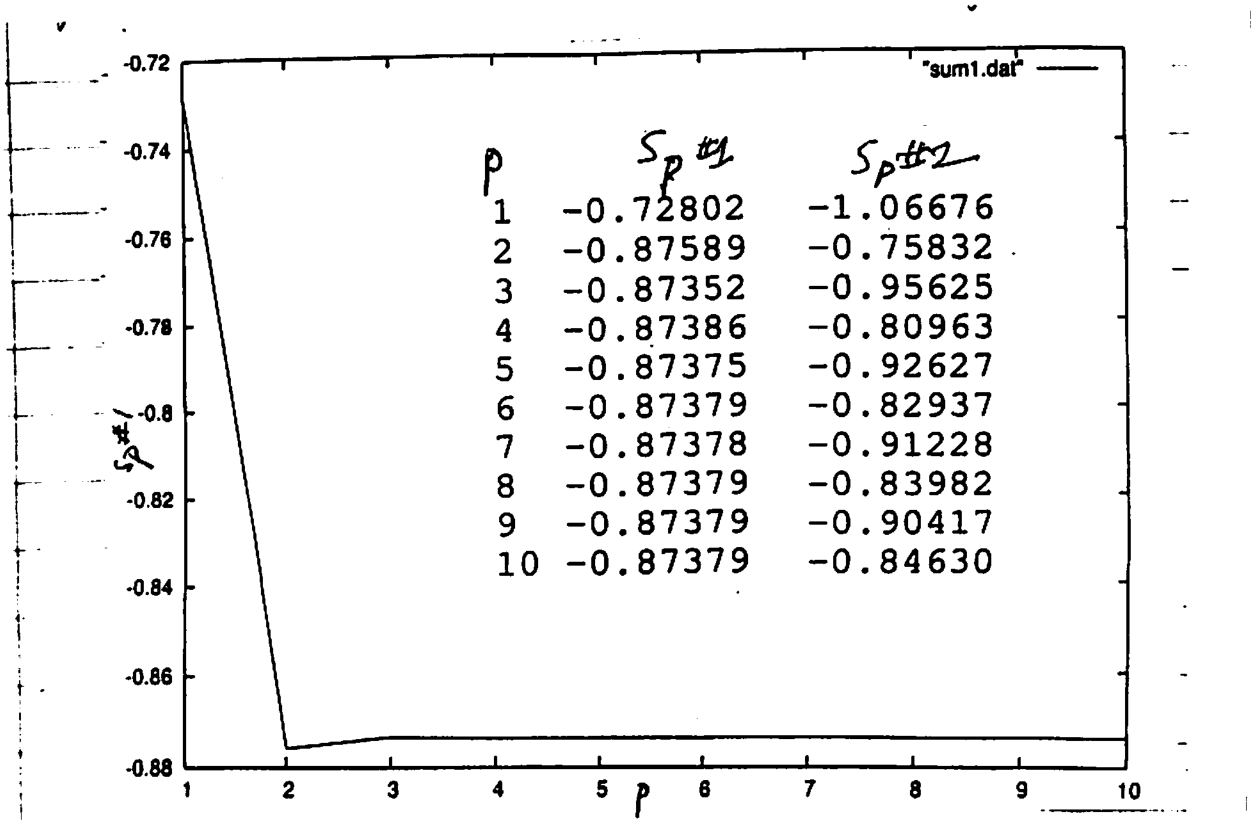
d) As

$$\begin{aligned} E(r) &= -\frac{Q}{4\pi\epsilon_0 r^2} \qquad a < r < b, \\ W_{tot} &= \frac{\epsilon_0}{2}\int E^2 d^3r, \\ &= \frac{Q^2}{8\pi\epsilon_0}\left[\frac{1}{a} - \frac{1}{b}\right] \end{aligned}$$

in agreement with c).

2) and 3) Below are plots of the partial sum vs. p in which the crystal is not neutral (a sum I called $Sp^{\#2}$) and in which it is neutral (a sum I called $Sp^{\#1}$. The difference in convergence is clear.





p	Sp #1	Sp #2
1	−0.72802	−1.06676
2	−0.87589	−0.75832
3	−0.87352	−0.95625
4	−0.87386	−0.80963
5	−0.87375	−0.92627
6	−0.87379	−0.82937
7	−0.87378	−0.91228
8	−0.87379	−0.83982
9	−0.87379	−0.90417
10	−0.87379	−0.84630

4) Let the charge $+q$ be at $\mathbf{r}_+ = \mathbf{r} + \mathbf{d}/2$ and the charge $-q$ be at $\mathbf{r}_- = \mathbf{r} - \mathbf{d}/2$ then the potential energy is

$$U = qV(\mathbf{r}_+) - qV(\mathbf{r}_-) = qV(\mathbf{r} + \mathbf{d}/2) - qV(\mathbf{r} - \mathbf{d}/2).$$

If $d << r$ then

$$\begin{aligned} U &= q[V(\mathbf{r}) + (\mathbf{d}/2)\cdot\nabla V(\mathbf{r}) + ..+] - q[V(\mathbf{r}) - (\mathbf{d}/2)\cdot\nabla V(\mathbf{r}) + ..+], \\ &= q\mathbf{d}\cdot\nabla V(\mathbf{r}) = -\mathbf{p}\cdot\mathbf{E}(\mathbf{r}). \end{aligned}$$

b) Let's take the origin of coordinates at the charge Q. Then

$$\mathbf{E} = \frac{Q}{4\pi\epsilon_0 r^2}\hat{\mathbf{r}}$$

and

$$U = -\mathbf{p}\cdot\mathbf{E} = -\frac{p\cos\theta}{4\pi\epsilon_0 r^2}$$

as before.

Problem 2.32

(a) $W = \frac{1}{2}\int \rho V\,d\tau$. From Prob. 2.21 (or Prob. 2.28): $V = \frac{\rho}{2\epsilon_0}\left(R^2 - \frac{r^2}{3}\right) = \frac{1}{4\pi\epsilon_0}\frac{q}{2R}\left(3 - \frac{r^2}{R^2}\right)$

$$W = \frac{1}{2}\rho\frac{1}{4\pi\epsilon_0}\frac{q}{2R}\int_0^R\left(3 - \frac{r^2}{R^2}\right)4\pi r^2 dr = \frac{q\rho}{4\epsilon_0 R}\left[3\frac{r^3}{3} - \frac{1}{R^2}\frac{r^5}{5}\right]\Big|_0^R = \frac{q\rho}{4\epsilon_0 R}\left(R^3 - \frac{R^3}{5}\right)$$
$$= \frac{q\rho}{5\epsilon_0}R^2 = \frac{qR^2}{5\epsilon_0}\frac{q}{\frac{4}{3}\pi R^3} = \boxed{\frac{1}{4\pi\epsilon_0}\left(\frac{3}{5}\frac{q^2}{R}\right).}$$

(b) $W = \frac{\epsilon_0}{2}\int E^2 d\tau$. *Outside* $(r > R)$ $\mathbf{E} = \frac{1}{4\pi\epsilon_0}\frac{q}{r^2}\hat{\mathbf{r}}$; *Inside* $(r < R)$ $\mathbf{E} = \frac{1}{4\pi\epsilon_0}\frac{q}{R^3}r\hat{\mathbf{r}}$.

$$\therefore W = \frac{\epsilon_0}{2}\frac{1}{(4\pi\epsilon_0)^2}q^2\left\{\int_R^\infty \frac{1}{r^4}(r^2 4\pi\,dr) + \int_0^R\left(\frac{r}{R^3}\right)^2(4\pi r^2 dr)\right\}$$
$$= \frac{1}{4\pi\epsilon_0}\frac{q^2}{2}\left\{\left(-\frac{1}{r}\right)\Big|_R^\infty + \frac{1}{R^6}\left(\frac{r^5}{5}\right)\Big|_0^R\right\} = \frac{1}{4\pi\epsilon_0}\frac{q^2}{2}\left(\frac{1}{R} + \frac{1}{5R}\right) = \frac{1}{4\pi\epsilon_0}\frac{3}{5}\frac{q^2}{R}.\checkmark$$

(c) $W = \frac{\epsilon_0}{2}\left\{\oint_{\mathcal{S}} V\mathbf{E}\cdot d\mathbf{a} + \int_{\mathcal{V}} E^2 d\tau\right\}$, where $\mathcal{V}$ is large enough to enclose all the charge, but otherwise arbitrary. Let's use a sphere of radius $a > R$. Here $V = \frac{1}{4\pi\epsilon_0}\frac{q}{r}$.

$$W = \frac{\epsilon_0}{2}\left\{\int_{r=a}\left(\frac{1}{4\pi\epsilon_0}\frac{q}{r}\right)\left(\frac{1}{4\pi\epsilon_0}\frac{q}{r^2}\right)r^2\sin\theta\,d\theta\,d\phi + \int_0^R E^2 d\tau + \int_R^a\left(\frac{1}{4\pi\epsilon_0}\frac{q}{r^2}\right)^2(4\pi r^2 dr)\right\}$$
$$= \frac{\epsilon_0}{2}\left\{\frac{q^2}{(4\pi\epsilon_0)^2}\frac{1}{a}4\pi + \frac{q^2}{(4\pi\epsilon_0)^2}\frac{4\pi}{5R} + \frac{1}{(4\pi\epsilon_0)^2}4\pi q^2\left(-\frac{1}{r}\right)\Big|_R^a\right\}$$
$$= \frac{1}{4\pi\epsilon_0}\frac{q^2}{2}\left\{\frac{1}{a} + \frac{1}{5R} - \frac{1}{a} + \frac{1}{R}\right\} = \frac{1}{4\pi\epsilon_0}\frac{3}{5}\frac{q^2}{R}.\checkmark$$

As $a \to \infty$, the contribution from the surface integral $\left(\frac{1}{4\pi\epsilon_0}\frac{q^2}{2a}\right)$ goes to zero, while the volume integral $\left(\frac{1}{4\pi\epsilon_0}\frac{q^2}{2a}\left(\frac{6a}{5R} - 1\right)\right)$ picks up the slack.