

PHYSICS 321; SOLUTION TO PROBLEM SET 3

1. The electric field on the axis of a ring of radius r and thickness dr and with a charge $dQ = \sigma 2\pi r dr$ is easily calculated to be

$$d\mathbf{E}(z) = \frac{1}{4\pi\epsilon_0} \frac{\sigma 2\pi r dr z}{(r^2 + z^2)^{3/2}} \hat{\mathbf{z}}$$

where z is the height above the plane of the ring. To get the field above the disk on the axis, we simply integrate this over r from 0 to R ;

$$\begin{aligned} \mathbf{E}(z) &= \frac{\sigma z}{2\epsilon_0} \int_0^R \frac{r}{(r^2 + z^2)^{3/2}} dr \hat{\mathbf{z}}, \\ &= \frac{\sigma}{2\epsilon_0} \left[1 - \frac{z}{(R^2 + z^2)^{1/2}} \right] \hat{\mathbf{z}} \end{aligned}$$

2. From the first problem we find that

$$\begin{aligned} E(z) &= \frac{\sigma}{2\epsilon_0} \left\{ \left[1 - \frac{z - d/2}{[R^2 + (z - d/2)^2]^{1/2}} \right] - \left[1 - \frac{z + d/2}{[R^2 + (z + d/2)^2]^{1/2}} \right] \right\}, \quad |z| > d/2 \\ &= \frac{\sigma}{2\epsilon_0} \left\{ \frac{z + d/2}{[R^2 + (z + d/2)^2]^{1/2}} - \frac{z - d/2}{[R^2 + (z - d/2)^2]^{1/2}} \right\}, \quad |z| > d/2 \\ &= -\frac{\sigma}{2\epsilon_0} \left\{ 2 - \frac{z - d/2}{[R^2 + (z - d/2)^2]^{1/2}} - \frac{z + d/2}{[R^2 + (z + d/2)^2]^{1/2}} \right\} \quad |z| < d/2. \end{aligned}$$

When $z \rightarrow d/2$ from above, we get

$$E(d/2+) = \frac{\sigma}{2\epsilon_0} \frac{d}{[R^2 + d^2]^{1/2}} \approx \frac{\sigma}{2\epsilon_0} \frac{d}{R} \quad \text{for } d \ll R.$$

3.a) Given $\rho = A/r$.

$$Q = \int \rho(r) d^3r = 4\pi \int_0^a \frac{A}{r} r^2 dr = 2\pi A a^2,$$

so $A = Q/(2\pi a^2)$.

Use Gauss' law. The electric field is in the radial direction, by symmetry. Outside

$$E_r(r) = \frac{Q}{4\pi\epsilon_0 r^2}, \quad r \geq a.$$

Inside, Gauss' law gives

$$\begin{aligned} 4\pi r^2 E_r(r) &= 4\pi \int_0^r A r dr = \frac{Q r^2}{\epsilon_0 a^2}, \quad \text{so} \\ E_r(r) &= \frac{Q}{4\pi\epsilon_0 a^2}, \quad r \leq a. \end{aligned}$$

and is constant inside.

4.) Given

$$\begin{aligned} \mathbf{E}(s) &= \frac{\lambda \hat{\mathbf{s}}}{2\pi\epsilon_0 s} \quad \text{then} \\ V(s) &= V(s_0) - \int \mathbf{E} \cdot d\mathbf{l} = V(s_0) - \int_{s_0}^s E(s) ds, \\ &= V(s_0) - \int_{s_0}^s \frac{\lambda}{2\pi\epsilon_0 s} ds, \\ &= -\frac{\lambda}{2\pi\epsilon_0} \ln\left(\frac{s_0}{s}\right) \end{aligned}$$

where I have taken $V(s_0) = 0$. This is one case where one can not take the potential to be zero at infinity.

5. Potential of a line of charge.

$$\begin{aligned} V(z, s) &= \frac{\lambda}{4\pi\epsilon_0} \int_{-a/2}^{a/2} \frac{dz'}{[(z' - z)^2 + s^2]^{1/2}}, \\ &= \frac{\lambda}{4\pi\epsilon_0} \int_{-(a/2+z)}^{a/2-z} \frac{dy}{[y^2 + s^2]^{1/2}}, \\ &= \frac{\lambda}{8\pi\epsilon_0} \ln \left\{ \left[\frac{\sqrt{(a/2 - z)^2 + s^2} + (a/2 - z)}{\sqrt{(a/2 - z)^2 + s^2} - (a/2 - z)} \right] \left[\frac{\sqrt{(a/2 + z)^2 + s^2} + (a/2 + z)}{\sqrt{(a/2 + z)^2 + s^2} - (a/2 + z)} \right] \right\}. \end{aligned}$$

a) What happens when $s \rightarrow 0$? Let us take $z > a/2$. Then

$$\begin{aligned}\sqrt{(a/2 - z)^2 + s^2} &= (z - a/2) \left[1 + \frac{s^2}{(z - a/2)^2} \right]^{1/2} \rightarrow z - a/2 + \frac{s^2}{2(z - a/2)}, \\ \sqrt{(a/2 - z)^2 + s^2} + (a/2 - z) &\rightarrow \frac{s^2}{2(z - a/2)} \quad \text{and} \\ \sqrt{(a/2 - z)^2 + s^2} - (a/2 - z) &\rightarrow 2(z - a/2). \quad \text{Also} \\ \sqrt{(a/2 + z)^2 + s^2} + (a/2 + z) &\rightarrow 2(z + a/2), \quad \text{and} \\ \sqrt{(a/2 + z)^2 + s^2} - (a/2 + z) &\rightarrow \frac{s^2}{2(z + a/2)}.\end{aligned}$$

Putting this all together we get

$$V(s, z) \rightarrow \frac{\lambda}{8\pi\epsilon_0} \ln \left[\frac{s^2(z + a/2)^2}{(z - a/2)^2 s^2} \right] = \frac{\lambda}{4\pi\epsilon_0} \ln \left[\frac{1 + a/2z}{1 - a/2z} \right], \quad a/2 < z.$$

As $V(s, z) = V(s, -z)$, we have

$$V(s, z) \rightarrow \frac{\lambda}{4\pi\epsilon_0} \ln \left[\frac{1 - a/2z}{1 + a/2z} \right] \quad z < -a/2.$$

b)

$$\begin{aligned}V(0, s) &= \frac{\lambda}{4\pi\epsilon_0} \ln \left\{ \frac{\left[\sqrt{(a/2)^2 + s^2} + (a/2) \right]}{\left[\sqrt{(a/2)^2 + s^2} - (a/2) \right]} \right\}, \\ &= \frac{\lambda}{4\pi\epsilon_0} \ln \left\{ \frac{\left[1 + (a/2s)^2 \right]^{1/2} + a/2s}{\left[1 + (a/2s)^2 \right]^{1/2} - a/2s} \right\}, \\ &\approx \frac{\lambda}{4\pi\epsilon_0} [\ln(1 + a/2s) - \ln(1 - a/2s)] \quad \text{for } a/2s \ll 1, \\ &\approx \frac{\lambda a}{4\pi\epsilon_0 s} \quad a/2s \ll 1,\end{aligned}$$

as it should.