SOLUTION TO PROBLEM SET 2

$$\begin{aligned}
\mathbf{v}(\mathbf{r}) &= \frac{\hat{\mathbf{i}} \times \mathbf{r}}{y^2 + z^2}, \\
&= \frac{\hat{\mathbf{i}} \times (x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}})}{y^2 + z^2}, \\
&= \frac{-z\hat{\mathbf{j}} + y\hat{\mathbf{k}}}{y^2 + z^2}
\end{aligned}$$

$$\begin{aligned}
\nabla \times \mathbf{v} &= \hat{\mathbf{i}} \left(\frac{\partial v_z}{\partial y} - \frac{\partial v_y}{\partial z} \right) + \hat{\mathbf{j}} \left(\frac{\partial v_x}{\partial z} - \frac{\partial v_z}{\partial x} \right) + \hat{\mathbf{k}} \left(\frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y} \right) \\
&= \hat{\mathbf{i}} \left(\frac{1}{y^2 + z^2} - \frac{2y^2}{(y^2 + z^2)^2} + \frac{1}{y^2 + z^2} - \frac{2z^2}{(y^2 + z^2)^2} \right), \\
&= \hat{\mathbf{i}} \left(\frac{2}{y^2 + z^2} - \frac{2}{y^2 + z^2} \right), \\
&= 0.
\end{aligned}$$

The above makes no sense on the x axis, of course, where the function diverges. But we can use Stoke's theorem. Let us take a circular path around the x axis of radius $(y^2 + z^2)^{1/2}$. Then an element of path length is

$$\mathbf{dl} = (-z\hat{\mathbf{j}} + y\hat{\mathbf{k}})d\theta$$

where θ is the angle around the x axis. Then $\mathbf{v} \cdot \mathbf{dl} = d\theta$ and we get

$$\oint \mathbf{v} \cdot \mathbf{dl} = \oint d\theta = 2\pi = \int (\nabla \times \mathbf{v}) \cdot d\mathbf{A}$$

Thus $\nabla \times \mathbf{v}$ must be singular on the x axis. As $\mathbf{dA}$ for an infinitesimal circle is certainly in the $\hat{\mathbf{k}}$ direction, we must have

$$\nabla \times \mathbf{v}(\mathbf{r}) = 2\pi\delta(y)\delta(z)\hat{\mathbf{k}}$$

2a)Note that the range of integration includes -2 where the delta function is singular.

$$\int_{-3}^{1}(x^3 - 3x^2 + 2x - 1)\delta(x+2)dx = (-2)^3 - 3(-2) + 2(-2) - 1 = -25$$

b) Again the range of integration includes π where the delta function is singular, so

$$\int_{0}^{\infty}[\cos(3x) + 2]\delta(x-\pi)dx = \cos(3\pi) + 2 = +1.$$

$$\delta(x^2 - a^2) = \delta[(x-a)(x+a)]$$

which is singular at $x = \pm a$. At $x = a$

$$\delta[(x-a)(x+a)] = \delta[(x-a)(2a)] = \delta(x-a)/2|a|$$

and similarly at $x = -a$

$$\delta[(x-a)(x+a)] = \delta[(x+a)(-2a)] = \delta(x+a)/2|a|$$

so quite generally

$$\delta[(x-a)(x+a)] = \frac{1}{2|a|}[\delta(x-a) + \delta(x+a)]$$

Problem 1.47

(a) $a^2 + \mathbf{a}\cdot\mathbf{a} + a^2 = \boxed{3a^2.}$

(b) $\int(\mathbf{r} - \mathbf{b})^2 \frac{1}{5^3}\delta^3(\mathbf{r})\, d\tau = \frac{1}{125}b^2 = \frac{1}{125}(4^2 + 3^2) = \boxed{\frac{1}{5}.}$

(c) $c^2 = 25 + 9 + 4 = 38 > 36 = 6^2$, **so c is outside** $\mathcal{V}$, **so the integral is** $\boxed{\text{zero.}}$

(d) $(\mathbf{e} - (2\,\hat{\mathbf{x}} + 2\,\hat{\mathbf{y}} + 2\,\hat{\mathbf{z}}))^2 = (1\,\hat{\mathbf{x}} + 0\,\hat{\mathbf{y}} + (-1)\,\hat{\mathbf{z}})^2 = 1 + 1 = 2 < (1.5)^2 = 2.25$, **so e is inside** $\mathcal{V}$, **and hence the integral is** $\mathbf{e}\cdot(\mathbf{d} - \mathbf{e}) = (3, 2, 1)\cdot(-2, 0, 2) = -6 + 0 + 2 = \boxed{-4.}$

4.a) In this case

$$\mathbf{p} = (-2\lambda)(1) + \int_{-1}^{1}[(-1)\hat{\imath} + y\hat{\jmath}]dy = -4\lambda\hat{\imath}$$

b) On the x axis,

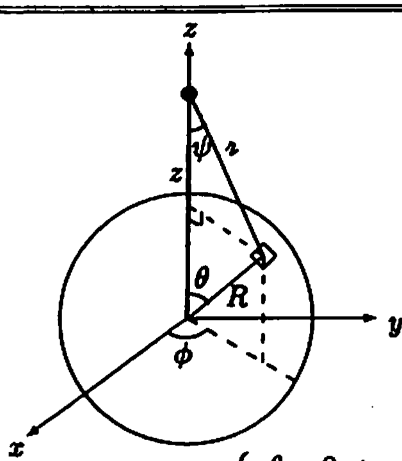
$$\begin{aligned}
\mathbf{E}(x,0) &= \hat{\imath}\left[-\frac{2\lambda}{4\pi\epsilon_0(x-1)^2} + \frac{\lambda}{4\pi\epsilon_0}\int_{-1}^{1}\frac{dy}{[(x+1)^2+y^2]}\frac{x+1}{[(x+1)^2+y^2]^{1/2}}\right]. \\
&= \hat{\imath}\frac{\lambda}{4\pi\epsilon_0}\left[-\frac{2}{(x-1)^2} + 2\int_0^1\frac{(x+1)dy}{[(x+1)^2+y^2]^{3/2}}\right], \\
&= \hat{\imath}\frac{2\lambda}{4\pi\epsilon_0}\left[-\frac{1}{(x-1)^2} + \frac{1}{(x+1)[(x+1)^2+1]^{3/2}}\right], \\
&= \hat{\imath}\frac{2\lambda}{4\pi\epsilon_0 x^2}\left[-\frac{1}{(1-x^{-1})^2} + \frac{1}{(1+x^{-1})[(1+x^{-1})^2+x^{-2}]^{1/2}}\right], \\
\rightarrow_{x>>1}\ & \hat{\imath}\frac{2\lambda}{4\pi\epsilon_0 x^2}\left[-(1+2x^{-1}) + (1-x^{-1})(1-x^{-1})\right], \\
&= \frac{2(-4\lambda\hat{\imath})}{4\pi\epsilon_0 x^3}
\end{aligned}$$

Now the electric field in the direction of a dipole $\mathbf{p}$ is

$$\mathbf{E} = \frac{2\mathbf{p}}{4\pi\epsilon_0 r^3},$$

which is just of the form of the above with a dipole moment $\mathbf{p} = -4\lambda\hat{\imath}$.

Problem 2.7



E is clearly in the z direction. From the diagram,

$dq = \sigma da = \sigma R^2 \sin\theta\, d\theta\, d\phi,$

$\imath^2 = R^2 + z^2 - 2Rz\cos\theta,$

$\cos\psi = \frac{z - R\cos\theta}{\imath}.$

So

$$\begin{aligned}
E_z &= \frac{1}{4\pi\epsilon_0}\int\frac{\sigma R^2\sin\theta\, d\theta\, d\phi(z - R\cos\theta)}{(R^2+z^2-2Rz\cos\theta)^{3/2}}. \qquad \int d\phi = 2\pi. \\
&= \frac{1}{4\pi\epsilon_0}(2\pi R^2\sigma)\int_0^{\pi}\frac{(z-R\cos\theta)\sin\theta}{(R^2+z^2-2Rz\cos\theta)^{3/2}}d\theta. \qquad \text{Let } u=\cos\theta;\ du = -\sin\theta\, d\theta;\ \left\{\begin{matrix}\theta = 0 \Rightarrow u = +1 \\ \theta = \pi \Rightarrow u = -1\end{matrix}\right\}. \\
&= \frac{1}{4\pi\epsilon_0}(2\pi R^2\sigma)\int_{-1}^{1}\frac{z-Ru}{(R^2+z^2-2Rzu)^{3/2}}du. \quad \text{Integral can be done by partial fractions—or look it up.} \\
&= \frac{1}{4\pi\epsilon_0}(2\pi R^2\sigma)\left[\frac{1}{z^2}\frac{zu-R}{\sqrt{R^2+z^2-2Rzu}}\right]_{-1}^{1} = \frac{1}{4\pi\epsilon_0}\frac{2\pi R^2\sigma}{z^2}\left\{\frac{(z-R)}{|z-R|} - \frac{(-z-R)}{|z+R|}\right\}.
\end{aligned}$$

For $z > R$ (outside the sphere), $E_z = \frac{1}{4\pi\epsilon_0}\frac{4\pi R^2\sigma}{z^2} = \frac{1}{4\pi\epsilon_0}\frac{q}{z^2}$, so $\boxed{\mathbf{E} = \frac{1}{4\pi\epsilon_0}\frac{q}{z^2}\hat{\mathbf{z}}.}$

For $z < R$ (inside), $E_z = 0$, so $\boxed{\mathbf{E} = 0.}$