Solution to Problem Set 1

1.

$$r'_i = \sum_i O_{ij} r_j.$$

so the square of the length is

$$\begin{aligned}
\sum_i r'_i r'_i &= \sum_{ijk} O_{ij} r_j O_{ik} r_k, \\
&= \sum_{jk} r_i r_j \sum_i O_{ij} O_{ik}, \\
&= \sum_{jk} r_j r_k \sum_i (O^T)_{ji} O_{ik}.
\end{aligned}$$

If we want the right-hand side to equal $\sum_j r_j r_j$, then we must require that

$$\begin{aligned}
\sum_i (O^T)_{ji} O_{ik} &= \delta_{jk} \qquad \text{or} \\
O^T O &= I, \qquad \text{then} \\
\sum_i r'_i r'_i &= \sum_{jk} r_j r_k \delta_{jk} = \sum_j r_j r_j
\end{aligned}$$

which is what we want.

2. Under inversion $\mathbf{A} \rightarrow -\mathbf{A}$ and $\mathbf{B} \rightarrow -\mathbf{B}$. Hence

$$\mathbf{A} \cdot \mathbf{B} \rightarrow (-\mathbf{A}) \cdot (-\mathbf{B}) = \mathbf{A} \cdot \mathbf{B}$$

On the other hand

$$\mathbf{C}' = \mathbf{A}' \times \mathbf{B}' \rightarrow (-\mathbf{A}) \times (-\mathbf{B}) = \mathbf{A} \times \mathbf{B} = \mathbf{C}$$

so that $\mathbf{C}' \rightarrow \mathbf{C}$, *not* $-\mathbf{C}$.

3.a)

$$\phi(x, y) = e^x \cos(y),$$

$$\begin{aligned}\nabla\phi(x,y) &= e^x\cos(y)\hat{\mathbf{i}} - e^x\sin(y)\hat{\mathbf{j}},\\ \nabla\phi(1,-\pi/4) &= \frac{e}{\sqrt{2}}\hat{\mathbf{i}} + \frac{e}{\sqrt{2}}\hat{\mathbf{j}}, \qquad \text{so the direction is}\\ \hat{\mathbf{r}} &= \frac{\hat{\mathbf{i}}+\hat{\mathbf{j}}}{\sqrt{2}} \qquad \text{and}\\ |\nabla\phi(1,-\pi/4)| &= e.\end{aligned}$$

b)

$$\nabla\phi(0,\pi/3) = \frac{\hat{\mathbf{i}} - \sqrt{3}\hat{\mathbf{j}}}{2}.$$

A unit vector in the direction given is

$$\hat{\mathbf{s}} = \frac{\hat{\mathbf{i}} + \sqrt{3}\hat{\mathbf{j}}}{2}$$

so that

$$\nabla\phi(0,\pi/3)\cdot\hat{\mathbf{s}} = \frac{1}{4} - \frac{3}{4} = -\frac{1}{2}$$

c)

$$\nabla\phi(0,\pi) = -\hat{\mathbf{i}}$$

so the direction is $-\hat{\mathbf{i}}$ and the magnitude is unity.

$$\begin{aligned}\nabla\phi(x,y,z) &= \nabla\frac{1}{(x^2+y^2+z^2)^{1/2}},\\ &= \left(\hat{\mathbf{i}}\frac{\partial}{\partial x} + \hat{\mathbf{j}}\frac{\partial}{\partial y} + \hat{\mathbf{k}}\frac{\partial}{\partial z}\right)\frac{1}{(x^2+y^2+z^2)^{1/2}},\\ &= -\frac{x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}}}{(x^2+y^2+z^2)^{3/2}},\\ &= -\frac{\mathbf{r}}{r^3} = -\frac{\hat{\mathbf{r}}}{r^2}.\end{aligned}$$

In spherical coordinates

$$\nabla\frac{1}{r} = \left(\hat{\mathbf{r}}\frac{\partial}{\partial r} + \hat{\theta}\frac{1}{r}\frac{\partial}{\partial\theta} + \hat{\phi}\frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}\right)\frac{1}{r}$$

$$= -\frac{\hat{\mathbf{r}}}{r^2}.$$

5.

$$\begin{aligned}
\mathbf{v} &= e^{kx}\sin(ky)\hat{\mathbf{i}} + e^{kx}\cos(ky)\hat{\mathbf{j}},\\
\nabla\cdot\mathbf{v} &= \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z},\\
&= ke^{kx}\sin(ky) - ke^{kx}\sin(ky) = 0.
\end{aligned}$$

Because $v_z = 0$ and v_x and v_y only depend upon x and y, the only component of $\nabla\times\mathbf{v}$ which is not obviously zero is

$$(\nabla\times\mathbf{v})_z = \frac{\partial v_x}{\partial y} - \frac{\partial v_y}{\partial x} = ke^{kx}\cos(ky) - ke^{kx}\cos(kx) = 0,$$

so $\nabla\times\mathbf{v} = 0$.

6.

$$\begin{aligned}
\phi &= \frac{\mathbf{p}\cdot\mathbf{r}}{r^3} = \frac{p_x x + p_y y + p_z z}{(x^2+y^2+z^2)^{3/2}},\\
\nabla\phi &= \left(\frac{p_x}{r^3} - 3\frac{p_x x^2}{r^5}\right)\hat{\mathbf{i}} + \left(\frac{p_y}{r^3} - 3\frac{p_y y^2}{r^5}\right)\hat{\mathbf{j}} + \left(\frac{p_z}{r^3} - 3\frac{p_z z^2}{r^5}\right)\hat{\mathbf{k}},\\
&= -3\frac{(\mathbf{p}\cdot\mathbf{r})\mathbf{r}}{r^5} + \frac{\mathbf{p}}{r^3},\\
&= -\frac{1}{r^3}[3(\mathbf{p}\cdot\hat{\mathbf{r}})\hat{\mathbf{r}} - \mathbf{p}].
\end{aligned}$$

When $\mathbf{p} = p\hat{\mathbf{k}}$, $\mathbf{p}\cdot\hat{\mathbf{r}} = p\cos\theta$, and

$$\nabla\phi = -\frac{p}{r^3}(3\cos\theta\hat{\mathbf{r}} - \hat{\mathbf{k}}).$$

This is plotted in your text in Fig. 5.55(a).

7. a)

$$\mathbf{F} = \hat{\mathbf{i}}\times\mathbf{r} = \hat{\mathbf{i}}\times(x\hat{i} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}} = -z\hat{\mathbf{j}} + y\hat{\mathbf{k}}.$$

Note that $\mathbf{F}$ is perpendicular to $\mathbf{r}$ (use $\mathbf{r} \cdot \mathbf{F} = 0$) and that $|F| = (y^2 + z^2)^{1/2}$, the distance from the x axis. Thus on a circle in the yz plane, $|F| = a$ and $\mathbf{F} \cdot d\mathbf{l} = Fad\theta = a^2 d\theta$. Therefore the line integral around the circle is simply

$$\oint \mathbf{F} \cdot d\mathbf{l} = 2\pi a^2.$$

b) $\nabla \times \mathbf{F} = 2\hat{\mathbf{i}}$ by direct calculation, so

$$\int (\nabla \times \mathbf{F}) \cdot d\mathbf{A} = 2 \int \hat{\mathbf{i}} \cdot d\mathbf{A} = 2 \int dA = 2\pi a^2.$$

Problem 1.39

$$\begin{aligned} \nabla\cdot\mathbf{v} &= \frac{1}{r^2}\frac{\partial}{\partial r}(r^2\, r\cos\theta) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\, r\sin\theta) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}(r\sin\theta\cos\phi) \\ &= \frac{1}{r^2}\, 3r^2\cos\theta + \frac{1}{r\sin\theta}\, r\, 2\sin\theta\cos\theta + \frac{1}{r\sin\theta}\, r\sin\theta(-\sin\phi) \\ &= 3\cos\theta + 2\cos\theta - \sin\phi = 5\cos\theta - \sin\phi \end{aligned}$$

$$\int(\nabla\cdot\mathbf{v})d\tau = \int(5\cos\theta - \sin\phi)\, r^2\sin\theta\, dr\, d\theta\, d\phi = \int_0^R r^2\, dr \int_0^{\frac{\pi}{2}} \left[\int_0^{2\pi}(5\cos\theta - \sin\phi)\, d\phi\right] d\theta \sin\theta$$

$$\hookrightarrow 2\pi(5\cos\theta)$$

$$= \left(\frac{R^3}{3}\right)(10\pi)\int_0^{\frac{\pi}{2}} \sin\theta\cos\theta\ , d\theta$$

$$\hookrightarrow \left.\frac{\sin^2\theta}{2}\right|_0^{\frac{\pi}{2}} = \frac{1}{2}$$

$$= \boxed{\frac{5\pi}{3}R^3.}$$

Two surfaces—one the hemisphere: $d\mathbf{a} = R^2\sin\theta\, d\theta\, d\phi\, \hat{\mathbf{r}}$; $r = R$; $\phi : 0 \to 2\pi$, $\theta : 0 \to \frac{\pi}{2}$.

$\int \mathbf{v}\cdot d\mathbf{a} = \int(r\cos\theta)R^2\sin\theta\, d\theta\, d\phi = R^3\int_0^{\frac{\pi}{2}}\sin\theta\cos\theta\, d\theta\int_0^{2\pi}d\phi = R^3\left(\frac{1}{2}\right)(2\pi) = \pi R^3.$

other the flat bottom: $d\mathbf{a} = (dr)(r\sin\theta\, d\phi)(+\hat{\theta}) = r\, dr\, d\phi\, \hat{\theta}$ (here $\theta = \frac{\pi}{2}$). $r : 0 \to R$, $\phi : 0 \to 2\pi$.

$\int \mathbf{v}\cdot d\mathbf{a} = \int(r\sin\theta)(r\, dr\, d\phi) = \int_0^R r^2\, dr\int_0^{2\pi}d\phi = 2\pi\frac{R^3}{3}.$

Total: $\int \mathbf{v}\cdot d\mathbf{a} = \pi R^3 + \frac{2}{3}\pi R^3 = \frac{5}{3}\pi R^3.$ ✓

9.Consider the i'th component

$$
\begin{aligned}
[\nabla \times (\mathbf{A} \times \mathbf{B})]_i &= \sum_{jk} \epsilon_{ijk} \frac{\partial}{\partial x_j} (\mathbf{A} \times \mathbf{B})_k, \\
&= \sum_{jk} \sum_{lm} \epsilon_{ijk} \frac{\partial}{\partial x_j} \epsilon_{klm} A_l B_m, \\
&= \sum_{jlm} \left[\sum_k \epsilon_{ijk} \epsilon_{klm} \right] \frac{\partial}{\partial x_j} A_l B_m, \\
&= \sum_{jlm} \left[\delta_{il} \delta_{jm} - \delta_{im} \delta_{jl} \right] \frac{\partial}{\partial x_j} A_l B_m, \\
&= \sum_j \frac{\partial}{\partial x_j} A_i B_j - \frac{\partial}{\partial x_j} A_j B_i, \\
&= \sum_j B_j \frac{\partial}{\partial x_j} A_i + A_i \frac{\partial}{\partial x_j} B_j - B_i \frac{\partial}{\partial x_j} A_j - A_j \frac{\partial}{\partial x_j} B_i, \\
&= (\mathbf{B} \cdot \nabla) A_i + A_i (\nabla \cdot \mathbf{B}) - B_i (\nabla \cdot \mathbf{A}) - (\mathbf{A} \cdot \nabla) B_i.
\end{aligned}
$$

As this is true for the i'th component, it follows that

$$\nabla \times (\mathbf{A} \times \mathbf{B}) = (\mathbf{B} \cdot \nabla)\mathbf{A} + \mathbf{A}(\nabla \cdot \mathbf{B}) - \mathbf{B}(\nabla \cdot \mathbf{A}) - (\mathbf{A} \cdot \nabla)\mathbf{B}$$

which is the identity in the flyleaf of your book.