PHYSICS 321

Autumn 2009

PROBLEM SET 1

Due Wednesday April 8

Problems marked with an asterisk will be graded.

1. Consider the operation O, represented by the 3×3 matrix O_{ij} which takes r into r';

$$r'_i = \sum_{j=1}^{3} O_{ij} r_j.$$

Show that if we require that the length of r' to be the same as the length of r, then $O^T O = I$, or

$$\sum_i O^T_{ji} O_{ik} = \delta_{jk}$$

where O^T is the transpose of O. One also easily shows that $OO^T = I$. Transformations which satisfy the requirements $O^T O = OO^T = I$ are called orthogonal transformations. All that we are requiring is that the transformation preserve the length of r.

2. * We defined a three-dimensional vector as any object with three components which transformed like the coordinates; i.e.

$$A'_i = \sum_{j=1}^{3} O_{ij} A_j,$$

where the operation O represents any translation or "proper" rotation. What we mean by "proper" is that it takes a right-handed coordinate system into a right-handed coordinate system. An "improper" rotation takes a right-handed system into a left-handed one. A reflection in a mirror does that, as does an inversion $\mathbf{r}' = O_{inv}\mathbf{r} = -\mathbf{r}$.

Let $\mathbf{A}$ and $\mathbf{B}$ be vectors. Under an inversion, $\mathbf{A} \rightarrow -\mathbf{A}$ and similarly for $\mathbf{B}$. Show that under an inversion, the dot product does not change. Show that the cross product, $\mathbf{C} = \mathbf{A} \times \mathbf{B}$ does *not* behave like a vector under this transformation. An object which behaves like a vector only under proper rotations is called a pseudovector. Note that as the position and momentum are true vectors, the angular momentum is a pseudo vector.

3.* Let $\phi = e^x \cos y$. Find

a) The direction in which ϕ is increasing most rapidly at $(1, -\pi/4)$ and the magnitude of the rate of increase.

b) The rate of change of ϕ with distance at $(0, \pi/3)$ in the direction $\hat{\mathbf{i}} + \hat{\mathbf{j}}\sqrt{3}$.

c) The direction and magnitude of $\nabla\phi$ at $(0, \pi)$.

4 Calculate the gradient of $(x^2 + y^2 + z^2)^{-1/2}$ in Cartesian coordinates and of r^{-1} in spherical coordinates.

5* Given

$$\mathbf{v} = e^{kx} \sin(ky)\hat{\mathbf{i}} + e^{kx} \cos(ky)\hat{\mathbf{j}},$$

verify that $\nabla \cdot \mathbf{v} = 0$ and $\nabla \times \mathbf{v} = 0$. Note, however, that $\mathbf{v}$ does not vanish at minus infinity.

6. Calculate $\nabla\Phi$ where $\Phi(\mathbf{r}) = \mathbf{p} \cdot \mathbf{r}/r^3$. Sketch $\nabla\Phi$ for $\mathbf{p} = p\hat{\mathbf{k}}$

7*. Consider the function $F(\mathbf{r}) = \hat{\mathbf{i}} \times \mathbf{r}$.

(a) Calculate directly the line integral of $F(\mathbf{r})$ around a circle in the yz plane centered at 0 with radius a.

(b) Calculate the line integral by using Stoke's theorem.

8*. Griffiths 1.39

9.* Use the totally antisymmetric tensors ϵ_{ijk} etc. to prove vector identity (8) in the flyleaf of Griffiths.