SOLUTION TO 321 FINAL EXAM SPRING 2009

1. a) This is very similar to problem 3 of the first hour exam. The electric field on the axis is clearly in the z direction. Its magnitude is obtained from

$$
\begin{aligned}
E_z(z) &= \frac{2\pi R\sigma}{4\pi\epsilon_0}\int_{-L/2}^{L/2}\frac{(z-z')}{[(z-z')^2+R^2]^{3/2}}dz', \\
&= \frac{R\sigma}{2\epsilon_0}\left[\frac{1}{[(z-z')^2+R^2]^{1/2}}\right]_{-L/2}^{L/2}, \\
&= \frac{R\sigma}{2\epsilon_0}\left\{\frac{1}{[(z-L/2)^2+R^2]^{1/2}}-\frac{1}{[(z+L/2)^2+R^2]^{1/2}}\right\}
\end{aligned}
$$

b)

$$E_z(L/2)=\frac{\sigma}{2\epsilon_0}\left[1-\frac{1}{[1+(L/R)^2]^{1/2}}\right]$$

c) Set $eE_z(L/2)=e^2/4\pi\epsilon_0 L^2$ to obtain

$$
\begin{aligned}
\frac{\sigma}{(e/2\pi\epsilon_0 LR)} &= \frac{R}{L}\frac{1}{[1-1/[1+(L/R)^2]^{1/2}]}, \\
&= \frac{R}{L}\frac{1}{[1-(R/L)/[1-(R/L)^2]^{1/2}]}
\end{aligned}
$$

d) For the numbers given, R/L=0.2 so that

$$\frac{\sigma}{(e/2\pi\epsilon_0 LR)}=\frac{0.2}{1-0.2/[1-(0.2)^2]^{1/2}}=0.25$$

so that there is the equivalent of 1/4 of a charge smeared over the surface of the channel. Of course if the charge is less than this, it is even easier for one K^+ to be pushed out of the channel by another.

2. The energy of the system, including the battery is

$$U = -\frac{1}{2}\frac{\kappa\epsilon_0 A}{d}V^2,$$

so the force is

$$\begin{aligned} F &= -\frac{\partial U}{\partial d}, \\ &= -\frac{\kappa\epsilon_0 AV^2}{2d^2}, \qquad \text{and the magnitude of the pressure is} \\ p &= |\frac{F}{A}| = \frac{\kappa\epsilon_0 V^2}{2d^2}, \qquad \text{so} \\ d &= \left(\frac{\kappa\epsilon_0 V^2}{2p}\right)^{1/2} \\ &= \left(\frac{(3.2)(8.85\times 10^{-12})(3\times 10^3)^2}{4\times 10^5}\right)^{1/2}, \\ &= 25\times 10^{-6} m, \end{aligned}$$

or 25 microns which is the right order of magnitude for mylar films.

3. a) We know from Gauss law and symmetry that $E_r(r) = A/r$. To evaluate the constant A

$$V_I - V_0 = -\int_{R_O}^{R_I} E(r)dr = -A\int_{R_O}^{R_I} dr = -A\ln(R_I/R_O),$$

so

$$A = (V_I - V_O)/\ln(R_O/R_I)$$

and

$$E_r(r) = \frac{V_I - V_O}{r\ln(R_O/R_I)}$$

b) There are several ways to do this. One can calculate the energy stored in a length L from

$$\begin{aligned} U &= \frac{\kappa\epsilon_0}{2}2\pi L\int E(r)^2 r dr, \\ &= \frac{\kappa\epsilon_0}{2}2\pi L\frac{(V_I - V_O)^2}{(\ln(R_O/R_I)^2}\ln(R_O/R_I), \end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{2}C(V_0 - V_I)^2 \qquad \text{so that} \\
C &= \frac{\kappa\epsilon_0 2\pi L}{\ln(R_O/R_I)}
\end{aligned}
$$

For $R_O = R_I + d$ and $d << R_O, R_I$, $\ln(R_O/R_I) = \ln(1 + d/R_I) \approx d/R$, so that

$$
\begin{aligned}
C &= 2\pi LR\frac{\kappa\epsilon_0}{d}, \qquad \text{or} \\
C/A &= \frac{\kappa\epsilon_0}{d} = \frac{(2)(8.85 \times 10^{-12})}{5 \times 10^{-10}} = 35 \times 10^{-3} F/m^2.
\end{aligned}
$$

An alternate way is to use the fact that $D = \kappa\epsilon_0 E$, so that the free charge per unit area at the outer radius is $\kappa\epsilon_0 A/R_O$ and the total free charge on the outer cylinder in length L is $Q_f = 2\pi R_O L\kappa\epsilon_0 A/R_O = 2\pi L\kappa\epsilon_0(V_I - V_O)/\ln(R_O/R_I) = C(V_I - V_O)$. This gives the same expression for the capacitance as before.

4. a) The closest the atoms can get is $2R$ so the energy to separate them is

$$
U_{vac} = \int_{2R}^{\infty} \frac{q^2}{4\pi\epsilon_0 r^2} dr = \frac{q^2}{4\pi\epsilon_0(2R)}.
$$

b) In water, the analogous result is

$$
U_{water} = \int_{2R}^{\infty} \frac{q^2}{4\pi\kappa\epsilon_0 r^2} dr = \frac{q^2}{4\pi\kappa\epsilon_0(2R)}.
$$

c)

$$
U_{water} = \frac{(1.6 \times 10^{-19})^2(9 \times 10^9)}{(80)(2 \times 10^{-10})} = 1.4 \times 10^{-20} J \approx 3.5 k_B T
$$

at room temperature, so an atom is easily thermally disassociated.

d)

$$
\begin{aligned}
f_{water} &= \exp(-U_{water}/k_B T), \\
f_{vac} &= \exp(-U_{vac}/k_B T) = \exp(-\kappa U_{water}/k_B T), \qquad \text{so} \\
f_{vac} &= f_{water}^{\kappa}
\end{aligned}
$$

5. a) For $r \leq R_1$, the form of the potential is

$$V(r,\theta) = \sum_l A_l r^l P_l(\cos\theta).$$

Because $P_1(\cos\theta) = \cos\theta$, it is clear that all $A_l = 0$ with the exception of $A_1 = V_0/R_1$ so that

$$V(r,\theta) = V_0 \frac{r}{R_1} \cos\theta, \qquad r \leq R_1.$$

b) Similarly for $r \geq R_2$, the form of the potential is

$$V(r,\theta) = \sum_l \frac{B_l}{r^{l+1}} P_l(\cos\theta),$$

and it is clear that all $B_l = 0$ except for $B_1 = -V_0 R_2^2$ so that

$$V(r,\theta) = -V_0 \frac{R_2^2}{r^2} \cos\theta, \qquad r \geq R_2.$$

For the region between the spheres, the form of the potential is

$$V(r,\theta) = \sum \left(C_l r^l + \frac{D_l}{r^{l+1}} \right) P_l(\cos\theta)$$

The equations needed to determine the coefficients are

$$\begin{aligned} V(R_1,\theta) &= V_0 P_1(\cos\theta) = \sum_l \left(C_l (R_1)^l + \frac{D_l}{(R_1)^{l+1}} \right) P_l(\cos\theta), \\ V(R_2,\theta) &= -V_0 P_1 \cos\theta = \sum_l \left(C_l (R_2)^l + \frac{D_l}{(R_2)^{l+1}} \right) P_l(\cos\theta). \end{aligned}$$

It should be clear that all $C_l = D_l = 0$ except for C_1 and D_1 so that the form of the potential is

$$V(r,\theta) = \left(C_1 r + \frac{D_1}{r^2} \right) P_1(\cos\theta).$$

with the two unknown coefficients determined by the two above equations.

6 a) Because $V(x,y) = V(-x,y) = V(x,-y)$, and because $V = 0$ on the left and right walls, I can certainly choose

$$V(x,y) = \sum_n A_n \cos kx \cosh ky.$$

The requirement $V(a/2,y) = 0$ implies that $k = (2l+1)\pi/a$ with $l = 0,1,2...$ One can just put this back into the above equations. Personally I would redefine the unknown coefficients so that the expansion reads

$$V(x,y) = \sum_{l=0}^{\infty} B_n \left(\frac{2}{a}\right)^{1/2} \cos[(2l+1)\pi x/a] \frac{\cosh[(2l+1)\pi y/a]}{\cosh[(2l+1)\pi/2]}$$

b) At $y = a/2$ we have from the above

$$1 = \sum_l B_l \left(\frac{2}{a}\right)^{1/2} \cos[(2l+1)\pi x/a]$$

from which we obtain, after multiplying by $(2/a)^{1/2}\cos[(2m+1)\pi x/a]$ and integrating from $-a/2$ to $a/2$,

$$\begin{aligned}
B_m &= \left(\frac{2}{a}\right)^{1/2} \int_{-a/2}^{a/2} \cos[(2m+1)\pi x/a] dx, \\
&= 2\left(\frac{2}{a}\right)^{1/2} \frac{a}{(2m+1)\pi} \sin[(2m+1)\pi/2), \\
&= (-1)^m \left(\frac{2}{a}\right)^{1/2} \frac{2a}{(2m+1)\pi} \qquad \text{so that} \\
V(x,y) &= \frac{4}{\pi} \sum_{l=0}^{\infty} \frac{(-1)^l}{2l+1} \cos[(2l+1)\pi x/a] \frac{\cosh[(2l+a)\pi y/a]}{\cosh[(2l+1)\pi/2]}.
\end{aligned}$$

THAT'S ALL FOLKS! ENJOY YOUR SUMMER. Michael Schick