

SOLUTION TO FIRST HOUR EXM

1. a)

$$\mathbf{E} = (2x^2 - 2xy - 2y^2) \hat{\mathbf{i}} + (-x^2 - 4xy + y^2) \hat{\mathbf{j}} = -\nabla V$$

implies

$$\begin{aligned} -\frac{\partial V}{\partial x} &= 2x^2 - 2xy - 2y^2 && \text{so that} \\ V(x, y) &= -\frac{2}{3}x^3 + x^2y + 2y^2x + f(y) && \text{where } f(y) \text{ is unknown} \\ -\frac{\partial V}{\partial y} &= -x^2 - 4xy + y^2, && \text{so that} \\ V(x, y) &= x^2y + 2xy - \frac{y^3}{3} + g(x) && \text{where } g(x) \text{ is unknown.} \end{aligned}$$

Comparing the two expressions for $V(x, y)$ we see that

$$V(x, y) = -\frac{x^3}{3} + x^2y + 2xy^2 - \frac{y^3}{3} + c$$

where c is a constant.

b)

$$\begin{aligned} \nabla \cdot \mathbf{E} &= \frac{\partial}{\partial x}(2x^2 - 2xy - 2y^2) + \frac{\partial}{\partial y}(-x^2 - 4xy + y^2), \\ &= 4x - 2y - 4x + 2y = 0. \end{aligned}$$

This is a field for which $\nabla \times \mathbf{E} = \nabla \cdot \mathbf{E} = 0$. The Helmholtz theorem does not apply to this function because it does not vanish at infinity.

2 a) The electric field on the axis is clearly in the z direction. Its magnitude is obtained from

$$E_z = \frac{2\pi R\sigma}{4\pi\epsilon_0} \left[\int_0^{L/2} \frac{(z - z')dz'}{[(z - z')^2 + R^2]^{3/2}} - \int_{-L/2}^0 \frac{(z - z')dz'}{[(z - z')^2 + R^2]^{3/2}} \right],$$

$$E_z = \frac{R\sigma}{2\epsilon_0} \left\{ \left[\frac{1}{[(z-z')^2 + R^2]^{1/2}} \right]_0^{L/2} - \left[\frac{1}{[(z-z')^2 + R^2]^{1/2}} \right]_{-L/2}^0 \right\},$$

$$\mathbf{E} = \frac{R\sigma}{2\epsilon_0} \left\{ \frac{1}{[(z-L/2)^2 + R^2]^{1/2}} + \frac{1}{[(z+L/2)^2 + R^2]^{1/2}} - \frac{2}{(z^2 + R^2)^{1/2}} \right\} \hat{\mathbf{k}}$$

b) At $z = 0$ the above is

$$E_z = -\frac{\sigma}{\epsilon_0} \left\{ 1 - \frac{1}{[1 + (L/2R)^2]^{1/2}} \right\}$$

3. a) By symmetry, the dipole moment vanishes.

b)

$$U = -\frac{(4e)(2e)}{4\pi\epsilon_0 d} - \frac{(4e)(2e)}{4\pi\epsilon_0 d} + \frac{(2e)(2e)}{4\pi\epsilon_0 (2d)},$$

$$= -14 \left(\frac{e^2}{4\pi\epsilon_0 d} \right).$$

$$c) \quad V(r, \theta) = \frac{1}{4\pi\epsilon_0} \left[\frac{4e}{r} - \frac{2e}{[(r \cos \theta - d)^2 + (r \sin \theta)^2]^{1/2}} - \frac{2e}{[(r \cos \theta + d)^2 + (r \sin \theta)^2]^{1/2}} \right],$$

$$= \frac{e}{4\pi\epsilon_0 r} \left[4 - \frac{2}{[1 - 2(d/r) \cos \theta + (d/r)^2]^{1/2}} - \frac{2}{[1 + 2(d/r) \cos \theta + (d/r)^2]^{1/2}} \right]$$

$$d) \quad [1 - [2(d \cos \theta / r) - (d/r)^2]]^{-1/2} = 1 + \frac{1}{2}[2(d \cos \theta / r) - (d/r)^2] - \frac{1}{8}(2d \cos \theta / r)^2 + \dots,$$

$$[1 + [2(d \cos \theta / r) + (d/r)^2]]^{-1/2} = 1 - \frac{1}{2}[2(d \cos \theta / r) + (d/r)^2] - \frac{1}{8}(2d \cos \theta / r)^2 + \dots,$$

so $V(r, \theta) \approx \frac{1}{4\pi\epsilon_0} \frac{4ed^2}{r^3} \frac{1 + \cos^2 \theta}{2}, \quad r \gg d.$