Quantum Mechanics (PHY 517)

Assignment 3 :

This problem set is due **Thursday October 18**, at the end of the lecture. Feel free to discuss the problems with others in the class, but you must write your own solutions.

1. Sakurai 8 : Given that $|+\rangle$ and $|-\rangle$ are the two eigenstates of the $\hat{S}_z$ operator acting on a spin $\frac{1}{2}$ system, determine

$$\left[\ \hat{S}_i, \hat{S}_j\ \right] \quad \text{and} \quad \{\ \hat{S}_i, \hat{S}_j\ \}$$

   where

$$\begin{aligned} \hat{S}_x &= \frac{\hbar}{2}\left(|+\rangle\langle -|\ +\ |-\rangle\langle +|\ \right) \\ \hat{S}_y &= i\frac{\hbar}{2}\left(|-\rangle\langle +|\ -\ |+\rangle\langle -|\ \right) \\ \hat{S}_z &= \frac{\hbar}{2}\left(|+\rangle\langle +|\ -\ |-\rangle\langle -|\ \right) \end{aligned}$$

2. Sakurai 9 : Construct the eigenstate of the $\mathbf{n}\cdot\hat{\mathbf{S}}$ operator with eigenvalue $+\frac{\hbar}{2}$, where $\mathbf{n}$ is the unit vector described by the usual spherical angles θ and ϕ. Start by constructing the 2×2 matrix that is $\mathbf{n}\cdot\hat{\mathbf{S}}$ in the basis where $\hat{S}_z$ is diagonal, then find its eigenvalues and eigenvectors.

3. Sakurai 11 : A two-state system has a Hamiltonian

$$\hat{H} \quad = \quad H_{11}|1\rangle\langle 1| + H_{22}|2\rangle\langle 2| + H_{12}\left(|1\rangle\langle 2| + |2\rangle\langle 1|\right)$$

   which is written in terms of the eigenstates, $|1\rangle$ and $|2\rangle$, of some operator that is not $\hat{H}$. Use the results of the previous question to find the energy eigenvalues and energy eigenstates of $\hat{H}$.

4. Sakurai 12 : A spin $\frac{1}{2}$ system is prepared in the eigenstate of the $\mathbf{n}\cdot\hat{\mathbf{S}}$ operator with eigenvalue $-\frac{\hbar}{2}$. The unit vector $\mathbf{n}$ is the same as in question 2.

(a) A measurement of S_x is made. What is the probability of measuring $+\frac{\hbar}{2}$?

(b) What is the dispersion in measurements of S_x on an ensemble of identically prepared systems,

$$\langle \left(\hat{S}_x - \langle \hat{S}_x \rangle \right)^2 \rangle$$

5. Sakurai 13 : A beam of ^3He nuclei (spin $\frac{1}{2}$) pass though a series of Stern-Gerlach-type magnets, with the following set up:

   (a) The first magnet is oriented in the $+z$ direction, and the $-\frac{\hbar}{2}$ component of the beam is directed into a beam dump.

   (b) The $+\frac{\hbar}{2}$ component passes through a second magnet aligned at an angle θ with respect to the z axis, rotated about the beam axis. The $-\frac{\hbar}{2}$ component leaving this magnet is directed into a beam dump.

   (c) The $+\frac{\hbar}{2}$ component passes through a third magnet aligned in the $+z$ direction.

   What is the ratio of ^3He nuclei with $-\frac{\hbar}{2}$ leaving this magnet compared with the number entering the first magnet?

6. Sakurai 14 : An operator in a three-dimensional ket space has a matrix representation

$$\hat{A} \quad \rightarrow \quad \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

   (a) What are the eigenvalues and eigenvectors of this matrix?

   (b) For what system could this be relevant?

7. Show that the double sum

$$\alpha \quad = \quad \sum_{\mathbf{K},\mathbf{N}} |\langle \mathbf{K}| \; \hat{B} \; |\mathbf{N}\rangle|^2$$

   is independent of the basis of kets $|\mathbf{N}\rangle$ and $|\mathbf{K}\rangle$