Electromagnetism 514

Final Examination 2006

Name (Please Print): ______________________________ Total Points: ______

(last) (first)

You should attempt all questions. There is a total of 100 points.

Problem 1	:	15 Points
Problem 2	:	20 Points
Problem 3	:	15 Points
Problem 4	:	20 Points
Problem 5	:	15 Points
Problem 6	:	15 Points

Formula that might be useful

$$\frac{1}{|\mathbf{x}-\mathbf{x}'|} = \sum_{m=-\infty}^{+\infty} \int_0^\infty dk\ e^{im(\phi-\phi')}\ J_m(k\rho)\ J_m(k\rho')\ e^{-k(z_>-z_<)} \quad .$$

$$\nabla(\mathbf{a}\cdot\mathbf{b}) = (\mathbf{a}\cdot\nabla)\mathbf{b} + (\mathbf{b}\cdot\nabla)\mathbf{a} + \mathbf{a}\times(\nabla\times\mathbf{b}) + \mathbf{b}\times(\nabla\times\mathbf{a}) \quad .$$

$$Y_{00} = \frac{1}{\sqrt{4\pi}} \quad .$$

$$Y_{11} = -\sqrt{\frac{3}{8\pi}}\ \sin\theta\ e^{i\phi} \quad , \quad Y_{10} = \sqrt{\frac{3}{4\pi}}\cos\theta \quad .$$

$$Y_{22} = \sqrt{\frac{15}{32\pi}}\sin^2\theta\ e^{i2\phi} \quad , \quad Y_{21} = -\sqrt{\frac{15}{8\pi}}\sin\theta\ \cos\theta\ e^{i\phi} \quad , \quad Y_{20} = \sqrt{\frac{5}{16\pi}}\left(3\cos^2\theta - 1\right) \quad .$$

$$Y_{L\ -M}(\theta,\phi) = (-)^m\ Y^*_{LM}(\theta,\phi) \quad .$$

$$\int_0^\infty dr\ r^\alpha\ e^{-\lambda r} = \frac{\alpha!}{\lambda^{\alpha+1}} \quad .$$

$$\nabla\cdot\mathbf{A} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 A_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta A_\theta) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}A_\phi$$

Final Examination - Electromagnetism (PHYS 514)

IMPORTANT: Answer ALL questions

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1. [**15 Total Points**] (Charge Distribution and Multipoles)

An isolated charge distribution has the form

$$\rho(r,\theta,\phi) \;=\; \frac{\lambda^5}{64\pi}\; r^2 e^{-\lambda r}\;\; \sin^2\theta \quad .$$

(a) [**8** points] Find the multipole moments, q_{LM}, of $\rho(r,\theta,\phi)$.

(b) [**7** points] Find the potential far from this charge distribution ($\lambda r \gg 1$) in terms of the spherical harmonics, $Y_{LM}(\theta,\phi)$.

2. [**20 Total Points**] (Currents, Permeable Material and Images)

Consider an isolated circular loop of radius a of fine wire carrying current I.

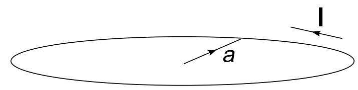

FIG. 1: *Current loop.*

(a) [**4** points] Show that the vector potential (in cylindrical coordinates) associated with this loop is $\mathbf{A} = A_\phi\ \hat{\mathbf{e}}_\phi$, with

$$A_\phi \ = \ \frac{\mu_0 I a}{2} \int_0^\infty dk\ J_1(k\ a)\ J_1(k\ \rho)\ e^{-k\ |z|} \quad .$$

Forgetting about the current loop for now, consider the situation where space is divided into two regions. The entire region with $z < 0$ is filled with a material of permeability μ, while the entire region $z \geq 0$ is vacuum.

(b) [**2** points] Starting with Maxwell's equations relevant to static charge and current distributions, derive the boundary conditions on the magnetic field that must be satisfied at the boundary at $z = 0$.

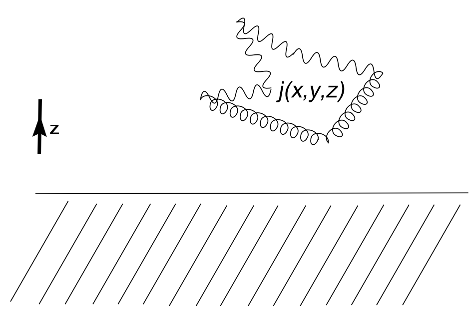

FIG. 2: *Current density* $\mathbf{j}(x, y, z)$ *above a semi-infinite permeable medium of permeability* μ.

(c) [**6** points] An arbitrary current density, $\mathbf{j}(x, y, z)$, is placed in the region $z > 0$. Find the image current distributions that can be used to determine the vector potential in both regions of space.

(d) [**6** points] Find the interaction energy between the circular current loop, considered in part (a), placed a distance d from the surface of the material with its plane parallel to the boundary (leave your answer as an integral over Bessel functions).

(e) [**2** points] What is the force on the loop (leave your answer as an integral over Bessel functions)

3. [**15 Total Points**] (Electric Potential)

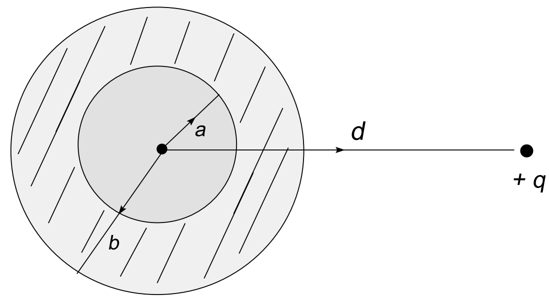

FIG. 3: *A grounded conducting sphere surrounded by dielectric near an isolated charge.*

A grounded conducting sphere of radius a is at the center of a uniform dielectric spherical shell of inner radius a and outer radius b, with dielectric constant ϵ. A charge $+q$ is placed a distance d from the center of the sphere.

(a) [**3** points] Write the form that the potential must have inside the dielectric as an expansion in Legendre polynomials or spherical harmonics.
(b) [**3** points] What are the boundary conditions that must be satisfied by the electric field at the dielectric-vacuum interface?
(c) [**9** points] What is the total charge on the conducting sphere?

4. [**20 Total Points**] (Magnetization)

The force experienced by a current distribution $\mathbf{j}$ in an external magnetic field $\mathbf{B}$ is

$$\mathbf{F} \;=\; \int d^3\mathbf{r}\; \mathbf{j}(\mathbf{r}) \;\times\; \mathbf{B}(\mathbf{r}) \quad .$$

(a) [**10** points] Show that a magnetized object, with magnetization density $\mathbf{M}(\mathbf{r})$, placed in an external magnetic field $\mathbf{B}$ experiences a force

$$\mathbf{F} \;=\; -\int d^3\mathbf{r}\; \mathbf{B}(\mathbf{r})\; \nabla\cdot\mathbf{M}(\mathbf{r}) \quad .$$

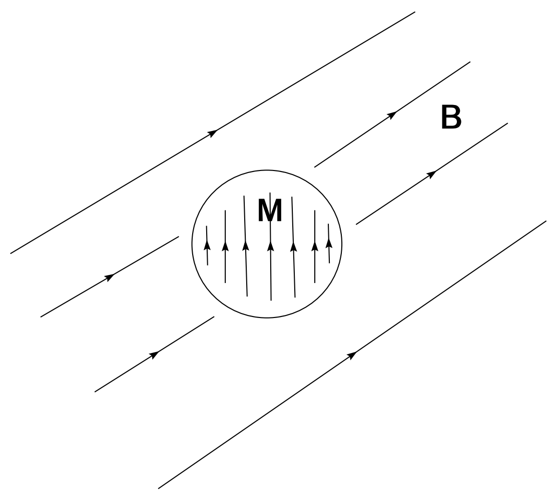

FIG. 4: *Uniformly magnetized sphere placed in a non-uniform magnetic field.*

(b) [**10** points] A unformly magnetized sphere aligned along the z-axis is placed in an external magnetic field of the form

$$\mathbf{B} \;=\; B_0\; [\; (1+\beta z)\hat{\mathbf{e}}_x \;+\; (1+\beta x)\hat{\mathbf{e}}_z \;] \quad .$$

What is the force on the sphere?

5. [**15 Total Points**] (Laplace)

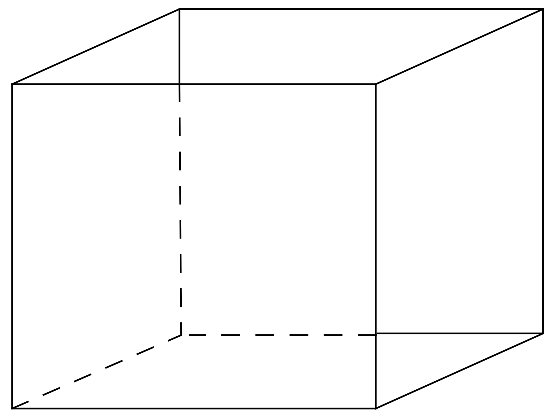

FIG. 5: *Cubic cavity with specified boundary conditions.*

Consider a cubic cavity with sides located at $x = 0, 1$, $y = 0, 1$ and $z = 0, 1$, i.e. the unit cube. The sides at $x = 0, 1$, $y = 0, 1$ are held at a potential $\Phi = 0$. The potentials on the side at $z = 0$ and $z = 1$ are

$$\Phi(x, y, z = 0) \;=\; -V_0 \sin(\pi x) \sin(2\pi y) \quad , \quad \Phi(x, y, z = 1) \;=\; V_0 \sin(2\pi x) \sin(\pi y) \quad ,$$

respectively. [**Notice** that the arguments of the sine-functions on the two surfaces are **not** the same.]

(a) [**12** points] Find the potential everywhere inside the cube.

(b) [**3** points] What are the maximum and minimum values of the potential and where are they located?

6. [**15 Total Points**] (Conductor and Dielectric)

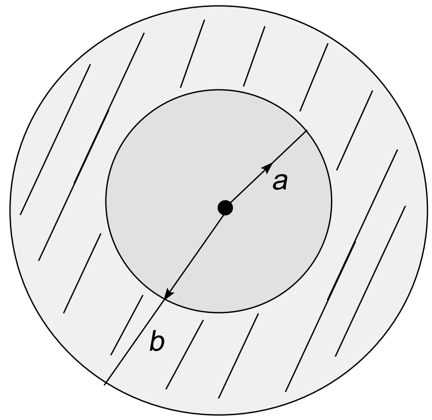

FIG. 6: *Cubic cavity with specified boundary conditions.*

An isolated conducting sphere of radius a is at the center of a dielectric spherical shell of inner radius a and outer radius b, with dielectric constant

$$\epsilon(\theta) \;=\; \epsilon_A \;+\; \epsilon_B \;\cos^2\theta \quad ,$$

with respect to the z-axis. A charge Q is placed on the conducting sphere.

(a) [**3** points] Why does the angular component of the electric displacement vanish everywhere?

(b) [**5** points] Find the electric field, electric displacement and polarization everywhere.

(c) [**3** points] What is the surface density of free-charge, and the total surface charge density?

(d) [**4** points] What is the capacitance of this system?