W Decay Kinematics and Constraints

Consider the decay of a $W^{\pm}$ into an e, ν pair, where we assume that both final leptons are massless and that we can measure the 3-vector of the charged lepton and the transverse momentum of the neutrino (from missing p_T). The relevant measured variables are

$$p_{z,e}, p_{T,e} : E_e = \sqrt{p_{z,e}^2 + p_{T,e}^2},$$
$$p_{T,\nu}, \Delta\phi_{e,\nu}.$$

The primary unknown is $p_{z,\nu}$, in terms of which we have

$$E_\nu = \sqrt{p_{z,\nu}^2 + p_{T,\nu}^2}.$$

Allowing the intermediate W to be offshell by an amount ΔM^2 we can use the onshell condition for the W to derive an expression for the unknown momentum,

$$\begin{aligned} 2p_e \cdot p_\nu &= M_W^2 + \Delta M^2 \\ &= 2\left(E_e E_\nu - p_{z,e} p_{z,\nu} - p_{T,e} p_{T,\nu} \cos\Delta\phi\right) \\ &= 2\left(E_e \sqrt{p_{z,\nu}^2 + p_{T,\nu}^2} - p_{z,e} p_{z,\nu} - p_{T,e} p_{T,\nu} \cos\Delta\phi\right). \end{aligned}$$

To simplify the notation let us give a label to the known transverse dot product (note the sign, we expect $M_T^2 \geq 0$, unless the W has extremely large transverse momentum so that the leptons are moving the same general direction, $\Delta\phi < \pi/2$),

$$M_T^2 \equiv -2p_{T,e} p_{T,\nu} \cos\Delta\phi.$$

Thus we can write

$$\sqrt{p_{z,\nu}^2 + p_{T,\nu}^2} = \left[p_{z,e} p_{z,\nu} + 0.5\left(M_W^2 + \Delta M^2 - M_T^2\right)\right] / E_e.$$

After squaring we have

$$p_{z,\nu}^2 + p_{T,\nu}^2 = \left[p_{z,e}^2 p_{z,\nu}^2 + p_{z,e} p_{z,\nu}\left(M_W^2 + \Delta M^2 - M_T^2\right) + 0.25\left(M_W^2 + \Delta M^2 - M_T^2\right)^2\right] / E_e^2$$

or

$$p_{z,\nu}^2 p_{T,e}^2 - p_{z,e} p_{z,\nu}\left(M_W^2 + \Delta M^2 - M_T^2\right) + p_{T,\nu}^2 E_e^2 - 0.25\left(M_W^2 + \Delta M^2 - M_T^2\right)^2.$$

Thus we can solve for the unknown longitudinal momentum of the neutrino,

$$\begin{aligned} p_{z,\nu} &= \frac{p_{z,e}\left(M_W^2 + \Delta M^2 - M_T^2\right) \pm \sqrt{p_{z,e}^2\left(M_W^2 + \Delta M^2 - M_T^2\right)^2 - 4p_{T,e}^2\left[p_{T,\nu}^2 E_e^2 - 0.25\left(M_W^2 + \Delta M^2 - M_T^2\right)^2\right]}}{2p_{T,e}^2} \\ &= \frac{p_{z,e}\left(M_W^2 + \Delta M^2 - M_T^2\right) \pm E_e\sqrt{\left(M_W^2 + \Delta M^2 - M_T^2\right)^2 - 4p_{T,e}^2 p_{T,\nu}^2}}{2p_{T,e}^2}. \end{aligned}$$

Since this expression describes the actual (real) kinematics we expect that

$$\left(M_W^2 + \Delta M^2 - M_T^2\right)^2 - 4p_{T,e}^2 p_{T,\nu}^2 \geq 0.$$

On the other hand, the usual constraint fit assumes an onshell W so that the implied longitudinal momentum is

$$\widetilde{p}_{z,\nu} = \frac{p_{z,e}\left(M_W^2 - M_T^2\right) \pm E_e\sqrt{\left(M_W^2 - M_T^2\right)^2 - 4p_{T,e}^2 p_{T,\nu}^2}}{2p_{T,e}^2}$$

and there is a chance that

$$\left(M_W^2 - M_T^2\right)^2 - 4p_{T,e}^2 p_{T,\nu}^2 < 0.$$

Using the previous (true) result we have

$$\left(M_W^2 - M_T^2\right)^2 - 4p_{T,e}^2 p_{T,\nu}^2 \geq -2\Delta M^2 \left(M_W^2 - M_T^2\right) - \Delta M^4.$$

Since we expect that $M_T^2 < M_W^2/2$ (the limit corresponds to transverse W decay at rest), the first term on the right is larger than the second and the sign of the right hand side depends on the sign of ΔM^2. Thus for a virtual W below the pole ($\Delta M^2 < 0$) the constraint should always yield a real solution. However, for an intermediate W above the pole ($\Delta M^2 > 0$) there should be a sizeable possibility of a complex solution, $\left(M_W^2 - M_T^2\right)^2 - 4p_{T,e}^2 p_{T,\nu}^2 < 0$.

Now let us compare this fake result to the true value for the minimum solutions to this equations,

$$p_{z,\nu\,\mathrm{min}} - \widetilde{p}_{z,\nu\,\mathrm{min}} = \frac{p_{z,e}\Delta M^2 + E_e\sqrt{\left(M_W^2 - M_T^2\right)^2 - 4p_{T,e}^2 p_{T,\nu}^2} - E_e\sqrt{\left(M_W^2 + \Delta M^2 - M_T^2\right)^2 - 4p_{T,e}^2 p_{T,\nu}^2}}{2p_{T,e}^2}.$$

Of particular interest is the case when the problems are just beginning

$$\left(M_W^2 - M_T^2\right)^2 - 4p_{T,e}^2 p_{T,\nu}^2 = 0,$$

$$p_{z,\nu\,\mathrm{min}} - \widetilde{p}_{z,\nu\,\mathrm{min}}\Big|_{\left(M_W^2 - M_T^2\right)^2 = 4p_{T,e}^2 p_{T,\nu}^2} = \frac{p_{z,e}\Delta M^2 - E_e\sqrt{2\Delta M^2\left(M_W^2 - M_T^2\right) + \Delta M^4}}{2p_{T,e}^2}.$$

This results suggests that $p_{z,\nu\,\mathrm{min}}$ will be smaller than $\widetilde{p}_{z,\nu\,\mathrm{min}}$ in this corner of phase space. This value of $\widetilde{p}_{z,\nu\,\mathrm{min}}$ is also what is returned by the rule that one should use the real part of the constrained value when it is complex. Let us assume that $\left(M_W^2 - M_T^2\right)^2 - 4p_{T,e}^2 p_{T,\nu}^2 < 0$ and use the real part of $\widetilde{p}_{z,\nu\,\mathrm{min}}$ to calculate an effective virtual W mass,

$$\begin{aligned}\widetilde{M}_W^2 &= 2p_e \cdot (\mathrm{Re}\,\widetilde{p}_\nu) = 2\left[E_e\sqrt{\left(\mathrm{Re}\,\widetilde{p}_{z,\nu}\right)^2 + p_{T,\nu}^2} - p_{z,e}\,\mathrm{Re}\,\widetilde{p}_{z,\nu} + 0.5M_T^2\right] \\ &= 2\left[\frac{E_e}{2p_{T,e}^2}\sqrt{p_{z,e}^2\left(M_W^2 - M_T^2\right)^2 + 4p_{T,e}^4 p_{T,\nu}^2} - \frac{p_{z,e}^2\left(M_W^2 - M_T^2\right)}{2p_{T,e}^2} + 0.5M_T^2\right].\end{aligned}$$

Now we want to eliminate the square root and try to arrange to replace most of the dependence on the 4-vector components with the true invariant mass. So isolate the square root and square,

$$\begin{aligned}\frac{E_e^2}{p_{T,e}^4}\left(p_{z,e}^2\left(M_W^2 - M_T^2\right)^2 + 4p_{T,e}^4 p_{T,\nu}^2\right) &= \left[\widetilde{M}_W^2 - M_T^2 + \frac{p_{z,e}^2\left(M_W^2 - M_T^2\right)}{p_{T,e}^2}\right]^2 \\ &= \frac{1}{p_{T,e}^4}\left[p_{T,e}^4\left(\widetilde{M}_W^2 - M_T^2\right)^2 + p_{z,e}^4\left(M_W^2 - M_T^2\right)^2 + 2p_{T,e}^2 p_{z,e}^2\left(M_W^2 - M_T^2\right)\left(\widetilde{M}_W^2 - M_T^2\right)\right].\end{aligned}$$

Manipulating and combining terms we find

$$\left(\widetilde{M}_W^2 - M_T^2\right)^2 + 2\frac{p_{z,e}^2\left(M_W^2 - M_T^2\right)\left(\widetilde{M}_W^2 - M_T^2\right)}{p_{T,e}^2} - \frac{p_{z,e}^2\left(M_W^2 - M_T^2\right)^2}{p_{T,e}^2} - 4E_e^2 p_{T,\nu}^2 = 0$$

and thus

$$\begin{aligned}\widetilde{M}_W^2 &= M_T^2 - \frac{p_{z,e}^2\left(M_W^2 - M_T^2\right)}{p_{T,e}^2} \pm \sqrt{\left(\frac{p_{z,e}^2\left(M_W^2 - M_T^2\right)}{p_{T,e}^2}\right)^2 + \frac{p_{z,e}^2\left(M_W^2 - M_T^2\right)^2}{p_{T,e}^2} + 4E_e^2 p_{T,\nu}^2} \\ &= \frac{p_{z,e}^2 M_W^2}{p_{T,e}^2} + \frac{E_e^2 M_T^2}{p_{T,e}^2} \pm \sqrt{\frac{E_e^2 p_{z,e}^2\left(M_W^2 - M_T^2\right)^2}{p_{T,e}^4} + 4E_e^2 p_{T,\nu}^2}\end{aligned}$$

Can this be simplified and/or related to ΔM^2?

Matt Bowen has proceeded by using the real part of the above expression, *i.e.*, define

$$\begin{aligned}
\widetilde{p}_{z,\nu} &= \mathrm{Re}\left[\frac{p_{z,e}\left(M_W^2 - M_T^2\right) \pm E_e\sqrt{\left(M_W^2 - M_T^2\right)^2 - 4p_{T,e}^2 p_{T,\nu}^2}}{2p_{T,e}^2}\right] \\
&= \frac{p_{z,e}\left(M_W^2 - M_T^2\right) \pm E_e\sqrt{\left(M_W^2 - M_T^2\right)^2 - 4p_{T,e}^2 p_{T,\nu}^2}}{2p_{T,e}^2} : \text{if } \left(M_W^2 - M_T^2\right)^2 - 4p_{T,e}^2 p_{T,\nu}^2 > 0, \\
&= \frac{p_{z,e}\left(M_W^2 - M_T^2\right)}{2p_{T,e}^2} : \text{if } \left(M_W^2 - M_T^2\right)^2 - 4p_{T,e}^2 p_{T,\nu}^2 < 0.
\end{aligned}$$

Another way to proceed is to explicitly use the expected Breit-Wigner shape of the W mass distribution and apply likelihood methods. Specifically we can define the trial mass as

$$\overline{M}_W^2\left(\overline{p}_{z,\nu}\right) \equiv 2p_e \cdot \overline{p}_\nu = 2\left[E_e\sqrt{\left(\overline{p}_{z,\nu}\right)^2 + p_{T,\nu}^2} - p_{z,e}\overline{p}_{z,\nu}\right] + M_T^2$$

and substitute this expression into a "likelihood" function given by the Breit-Wigner form

$$\mathcal{L}\left(\overline{p}_{z,\nu}\right) \equiv \frac{M_W^2\Gamma_W^2}{\left(M_W^2 - \overline{M}_W^2\left(\overline{p}_{z,\nu}\right)\right)^2 + M_W^2\Gamma_W^2},$$

where I am ignoring any issues about the $\overline{M}_W^2$ dependence of the width and the choice of normalization is for convenience. In general, when $\left(M_W^2 - M_T^2\right)^2 - 4p_{T,e}^2 p_{T,\nu}^2 > 0$ this function of $\overline{p}_{z,\nu}$ has two local maxima corresponding to $\overline{M}_W^2 = M_W^2$ with $\overline{p}_{z,\nu}$ values given by the two solutions of the above quadratic equations. For the other situation, $\left(M_W^2 - M_T^2\right)^2 - 4p_{T,e}^2 p_{T,\nu}^2 < 0$, the likelihood method requires that we solve for the (real) value of $\overline{p}_{z,\nu}$which maximizes this function, *i.e.*, yields a lepton pair mass as close as possible to M_W^2. This in turn arises from minimizing the quantity $\left[E_e\sqrt{\left(\overline{p}_{z,\nu}\right)^2 + p_{T,\nu}^2} - p_{z,e}\overline{p}_{z,\nu}\right]$, which happens for (the single value) $\overline{p}_{z,\nu} = p_{z,e}\left(p_{T,\nu}/p_{T,e}\right)$. Thus the summary of this method is to use the following neutrino longitudinal momenta

$$\begin{aligned}
\overline{p}_{z,\nu} &= \frac{p_{z,e}\left(M_W^2 - M_T^2\right) \pm E_e\sqrt{\left(M_W^2 - M_T^2\right)^2 - 4p_{T,e}^2 p_{T,\nu}^2}}{2p_{T,e}^2} : \text{if } \left(M_W^2 - M_T^2\right)^2 - 4p_{T,e}^2 p_{T,\nu}^2 > 0, \\
&= p_{z,e}\frac{p_{T,\nu}}{p_{T,e}} : \text{if } \left(M_W^2 - M_T^2\right)^2 - 4p_{T,e}^2 p_{T,\nu}^2 < 0.
\end{aligned}$$

Although it is not obvious, for the case of interest, $\left(M_W^2 - M_T^2\right)^2 - 4p_{T,e}^2 p_{T,\nu}^2 < 0$, $\widetilde{p}_{z,\nu}$ is essentially (numerically) identical to $\overline{p}_{z,\nu}$. Note, for example, that the two quantities are equal for the boundary case, $\left(M_W^2 - M_T^2\right)^2 - 4p_{T,e}^2 p_{T,\nu}^2 = 0$.