Algorithms and Kinematics

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We discuss various issues relevant to understanding differences in cone jet algorithms.

We want to compare the various versions of the cone jet variables applied to the perurbative calculation (and data). The primary issue is the way in which the 4-momenta of the two partons are combined to form a jet. To match the notation internal to the EKS code we will label the two partons as 2 and 3. The 4-momenta are then given by

$$p_j^\mu = (p_j \cosh y_j, p_j \cos\varphi_j, p_j \sin\varphi_j, p_j \sinh y_j), j = 2, 3.$$

Note that since the partons are massless there is no difference between rapidity and pseudorapidity. We assume that

$$p_3 = zp_2 \leq p_2$$

and define

$$\Delta\varphi \equiv \varphi_2 - \varphi_3, \Delta y \equiv y_2 - y_3.$$

The various algorithms are then defined by the following rules:

Scalar E_T (Snowmass): The jet E_T is given by the scalar sum of the individual scalar p_T's

$$E_{T,\mathrm{scalar}} = p_2 + p_3 = p_2 (1+z),$$

while the "effective" (pseudo)rapidity of the jet (*i.e.*, the cone) is given in terms of the weighted average variables,

$$y_J = \frac{p_2 y_2 + p_3 y_3}{p_2 + p_3} = \frac{y_2 + z y_3}{1+z}, \; \varphi_J = \frac{p_2\varphi_2 + p_3\varphi_3}{p_2 + p_3} = \frac{\varphi_2 + z\varphi_3}{1+z}.$$

The constraint for both partons to be in the cone of cone size R

$$(y_J - y_j)^2 + (\varphi_J - \varphi_j)^2 \leq R^2$$

can be expressed simply as

$$\Delta y^2 + \Delta\varphi^2 \leq (1+z)^2 R^2.$$

To account for configurations that might be missed in the experimental analyses due to the use of "seeds", the subsidiary constraint,

$$\Delta y^2 + \Delta\varphi^2 \leq R_{sep}^2 R^2,$$

with (typically) $R_{sep} = 1.3$, was also imposed. We will ignore this point in the discussion that follows immediately. Note that the Snowmass E_T variable has the strengths of being both simple and boost invariant (along the beam direction). However, even if the two partons could evolve into hadrons via showering and hadronization (they cannot due to color conservation), this scalar variable is not conserved, *i.e.*, does not correspond to an underlying symmetry.

4-D (or E-scheme) Kinematics: This is the choice advocated in the Run II jet study. In this case we use real 4-momenta definitions to construct the jet (keeping track of the invariant mass of the pair) to define the jet. Thus we have

$$p_J^\mu = (p_2\cosh y_2 + p_3 \cosh y_3, p_2\cos\varphi_2 + p_3\cos\varphi_3, p_2\sin\varphi_2 + p_3\sin\varphi_3, p_2\sinh y_2 + p_3\sinh y_3),$$

from which we can identify the jet p_T or E_T as

$$E_{T,4\ \mathrm{D}} = \sqrt{p_2^2 + p_3^2 + 2p_2 p_3 \cos(\varphi_2 - \varphi_3)} = E_{T,\mathrm{scalar}} \frac{\sqrt{1 + z^2 + 2z\cos(\varphi_2 - \varphi_3)}}{1+z} \leq E_{T,\mathrm{scalar}},$$

where the expression $(\varphi_2 - \varphi_3)$ refers to the minimum value of the separation in azimuth accounting for the periodic behavior of these variables. Hence the 4-D definition yields a *smaller* E_T value than Snowmass, with a difference that vanishes $(\varphi_2 - \varphi_3) \to 0$. Note that this definition for a jet E_T has the multiple virtues of being boost invariant and conserved (although the color coherence inherent in hadronization must bring in some local violation of E_T conservation). The corresponding 4-vector jet variables are given by

$$y_J = 0.5 \ln\left(\frac{p_2 e^{y_2} + p_3 e^{y_3}}{p_2 e^{-y_2} + p_3 e^{-y_3}}\right) = y_2 + 0.5 \ln\left(\frac{1 + z e^{-\Delta y}}{1 + z e^{\Delta y}}\right),$$
$$\varphi_J = \arctan\left(\frac{p_2 \sin\varphi_2 + p_3 \sin\varphi_3}{p_2 \cos\varphi_2 + p_3 \cos\varphi_3}\right) = -\arctan\left(\frac{z \sin\Delta\varphi}{1 + z\cos\Delta\varphi}\right),$$

where the final, simple expression corresponds to the choice $\varphi_2 = 0$, which we can always make. Using these expressions we can write the 4-D inequality for both partons in a jet as

$$\left(\Delta y + 0.5 \ln\left(\frac{1 + z e^{-\Delta y}}{1 + z e^{\Delta y}}\right)\right)^2 + \left(\Delta\varphi - \arctan\left(\frac{z \sin\Delta\varphi}{1 + z\cos\Delta\varphi}\right)\right)^2 \leq R^2.$$

CDF E_T: In Run I CDF used the 4-D definitions for defining jets plus defined a jet E_T that was more "E-like" than the p_T variable above. CDF used

$$E_{T,4\text{ D,CDF}} = p_J^0 \frac{\sqrt{(p_J^1)^2 + (p_J^2)^2}}{\sqrt{(p_J^1)^2 + (p_J^2)^2 + (p_J^3)^2}} = E_{T,4\text{ D}} \frac{p_J^0}{|\overrightarrow{p_J}|} = E_{T,4\text{ D}} \frac{\sqrt{|\overrightarrow{p_J}|^2 + M_J^2}}{|\overrightarrow{p_J}|} \geq E_{T,4\text{ D}}.$$

Thus the CDF definition yields an E_T that is larger than the 4-D definition,

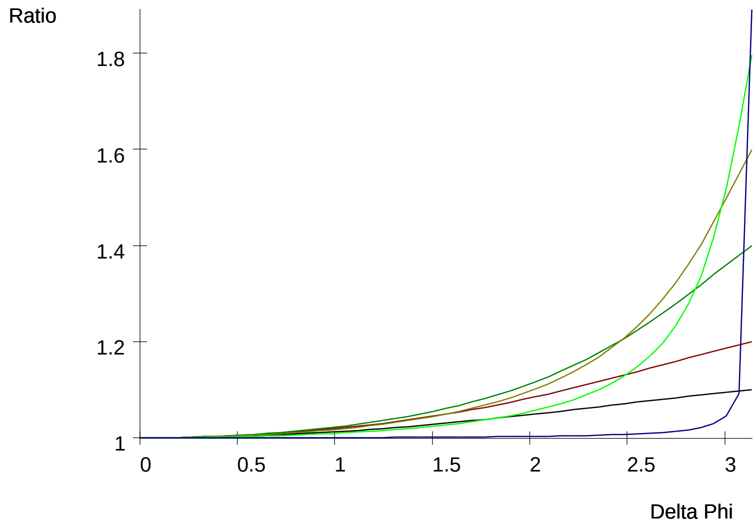
$$\frac{E_{T,4\text{ D,CDF}}}{E_{T,4\text{ D}}} = \frac{\sqrt{|\overrightarrow{p_J}|^2 + M_J^2}}{|\overrightarrow{p_J}|} \simeq 1 + \frac{M_J^2}{2|\overrightarrow{p_J}|^2},$$

by an amount that depends on the (nonzero) mass of the jet. Depending the relative importance of $M_J^2/|\overrightarrow{p_J}|^2$ and $(\varphi_2 - \varphi_3)$ corrections, the CDF value of E_T may be larger or smaller than the Snowmass value. Note finally that the CDF definition has the undesirable features of being neither boost invariant nor conserved.

To see how these different versions act let us first consider the phase space that contributes to the jet. We start with effectively 5 variables, $z, y_2, y_3, \varphi_2, \varphi_3$, which we can simplify to (ignoring the cuts on y) Δy and $\Delta\varphi$. First compare the Snowmass variables with the 4-D variables. For example, consider the ratio of the φ cuts, *i.e.*, the ratio of the distances of the lower energy parton from the cone center in the two cases,

$$R_\varphi(\Delta\varphi, z) = \frac{\Delta\varphi - \arctan\left(\frac{z\sin\Delta\varphi}{1+z\cos\Delta\varphi}\right)}{\left(\frac{\Delta\varphi}{1+z}\right)}.$$

$$\frac{x - \arctan\left(\frac{0.1\sin x}{1+0.1\cos x}\right)}{\left(\frac{x}{1+0.1}\right)}$$

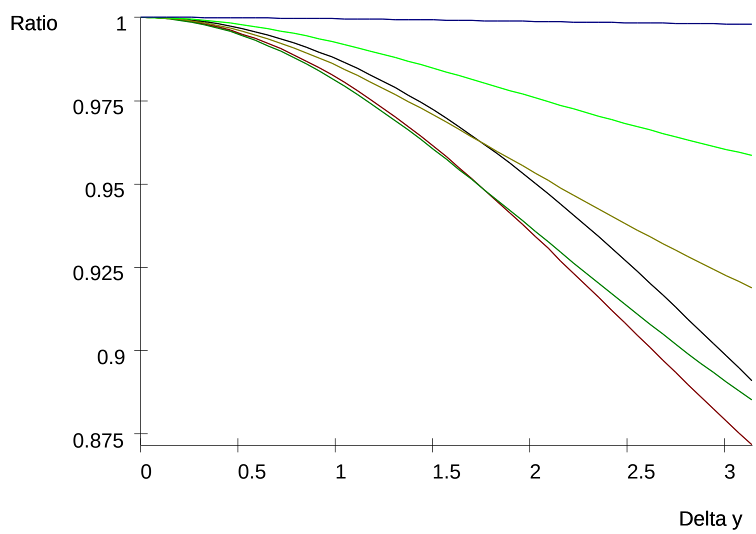


The ratio is plotted for z values of 0.1 (black), 0.2 (red), 0.4 (green), 0.6 (brown), 0.8 (light green), 0.99 (blue).

We see that the 4-D distance in φ is slightly larger and thus the two partons must be slightly closer together in the 4-D case in order to "fit" into a jet, *i.e.*, the phase space available to the 2-in-a-jet case is slightly smaller in the 4-D case. The corresponding rapidity separation ratio looks like

$$R_y\left(\Delta y, z\right) = \frac{\Delta y + 0.5 \ln\left(\frac{1+ze^{-\Delta y}}{1+ze^{\Delta y}}\right)}{\left(\frac{\Delta y}{1+z}\right)}.$$

$$\frac{x+0.5\ln\left(\frac{1+0.1e^{-x}}{1+0.1e^{x}}\right)}{\left(\frac{x}{1+0.1}\right)}$$



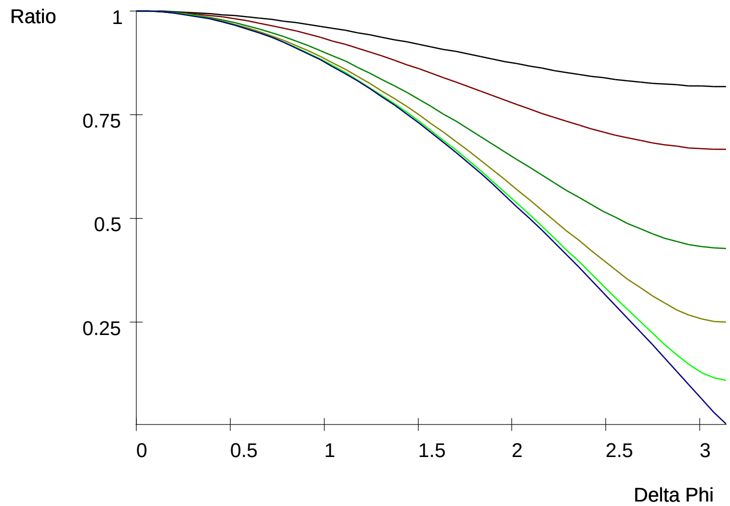
The ratio is plotted for z values of 0.1 (black), 0.2 (red), 0.4 (green), 0.6 (brown), 0.8 (light green), 0.99 (blue).

Hence in the case of rapidity the cut on the separation is slightly less restrictive in the 4-D case leading to slightly larger phase space. In practice the two effects tend to cancel and the jets, i.e., the 2-parton configurations allowed in a single jet are essentially identical in the 4-D and Snowmass algorithms.

A numerically more important question is the different choices of E_T variables with which the jets are labeled and binned. Consider first the ratio of the 4-D to Snowmass definitions,

$$R_{E_T}\left(\Delta\varphi, z\right) = \frac{\sqrt{1+z^2+2z\cos\left(\Delta\varphi\right)}}{1+z}.$$

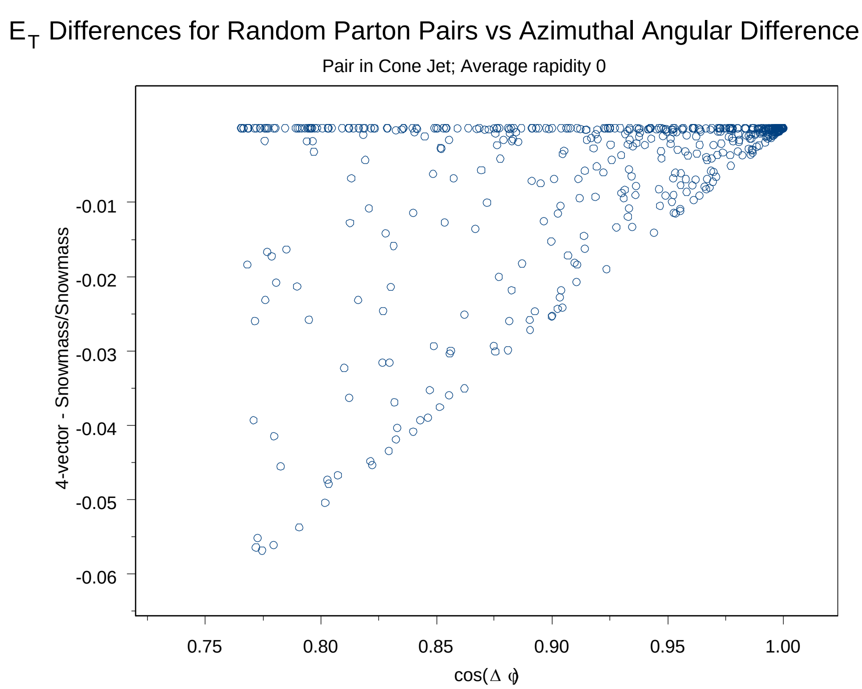
$$\frac{\sqrt{1+(0.1)^2+2(0.1)\cos(x)}}{1+0.1}$$



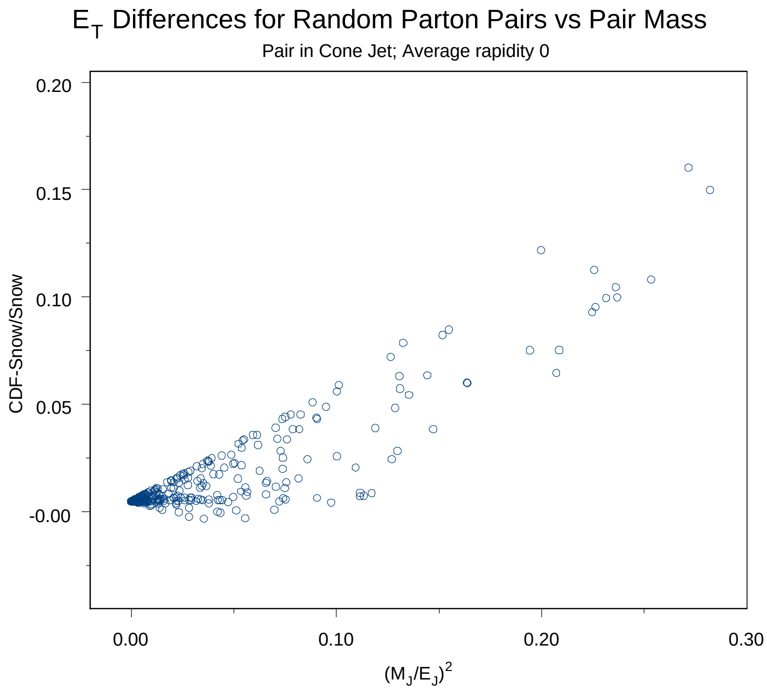
The ratio is plotted for z values of 0.1 (black), 0.2 (red), 0.4 (green), 0.6 (brown), 0.8 (light green), 0.99 (blue).

Thus, as expected, the same parton configurations will yield a smaller E_T value in the 4-D case than in the Snowmass definition.

To illustrate this in another fashion let us generate "random" pairs of partons, generated to yield a dE/E spectrum in energy but uniformly distributed in rapidity (around $y = 0$) and azimuth. For each pair of 4-vectors that can both "fit" in a jet by the rules given above (as noted above there is essentially no difference between the set of pairs allowed by the Snowmass definition and that allowed by the 4-vector definition) we can calculate the various E_T values and look at scatter plots. The first scatter plot exhibits $(E_{T,4\text{ D}} - E_{T,\text{scalar}})/E_{T,\text{scalar}}$ versus $\cos(\Delta\phi)$ for the pairs. This clearly illustrates the fact that $(E_{T,4\text{D}} - E_{T,\text{scalar}}) < 0$ as noted above. Those



pairs with $z \ll 1$ populate the upper boundary where there is no difference between the E_T's. We also see the expected correlation with $\cos(\Delta\phi)$. The characteristic size of the effect is expected to be of order a few %,



depending on the specific distribution in z and $\Delta\phi$. Note that this plot will not change if we boost all pairs by the same amount.

Consider next the comparison of the CDF definition with the Snowmass case. The next scatter plot shows $\left(E_{T,\mathrm{4D,\ CDF}} - E_{T,\mathrm{scalar}}\right)/E_{T,\mathrm{scalar}}$ versus $(M_J/E_J)^2$, which is suggested as the relevant quantity above. Note now, as expected, that the difference is almost always positive and of slightly larger magnitude. We also see the expected correlation between a larger difference and a larger invariant mass for the pair. To illustrate the lack of boost invariance we can boost all of the 4-vectors by $\Delta y = 2$ and recalculate the same quantities. The scatter plot now looks like the following. The most important point is that the sign of the effect has now changed for almost all of the pairs. For large rapidity and thus large energies the $M_J^2/\left|\overrightarrow{p_J}\right|^2$ effects have now almost vanished. The remaining differences arise from the 4-vector effects and are negative. Note that the abscissa has changed also because it is not invariant itself. If we plot versus $\cos(\Delta\phi)$, we see a result very similar to the 4-vector plot above.

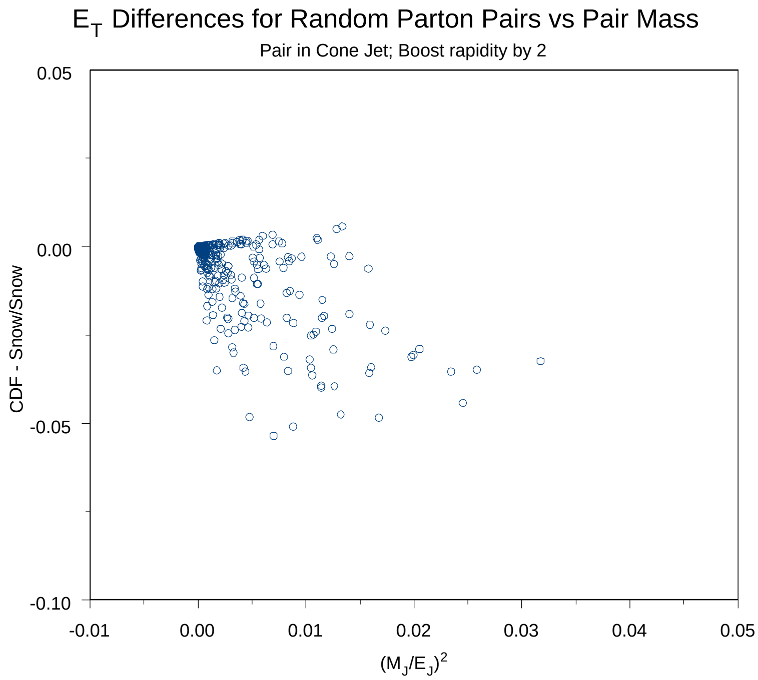
So how do these differences influence the jet cross sections themselves? Since the jet cross section is a rapidly falling function of E_T, the difference in the E_T variable described above will be translated directly into differences in the cross sections. The sign of the effect will be preserved, while the magnitude will be magnified by the large (falling) slope of the cross section. This point is illustrated by the following figure. For the CDF rapidity range, $0.1 < |y| < 0.7$, the CDF E_T definition yields the largest cross section and the 4-vector, or E-scheme, definition results in the smallest cross section. The latter approach yields a jet cross section smaller than the Snowmass case by about 4%, or slightly more (at the larger E_T values), with little other structure. At small rapidity the CDF definition cross section is larger than Snowmass by about 3%. At larger rapidity this relationship will change sign and eventually the CDF cross section will approach the 4-D result.

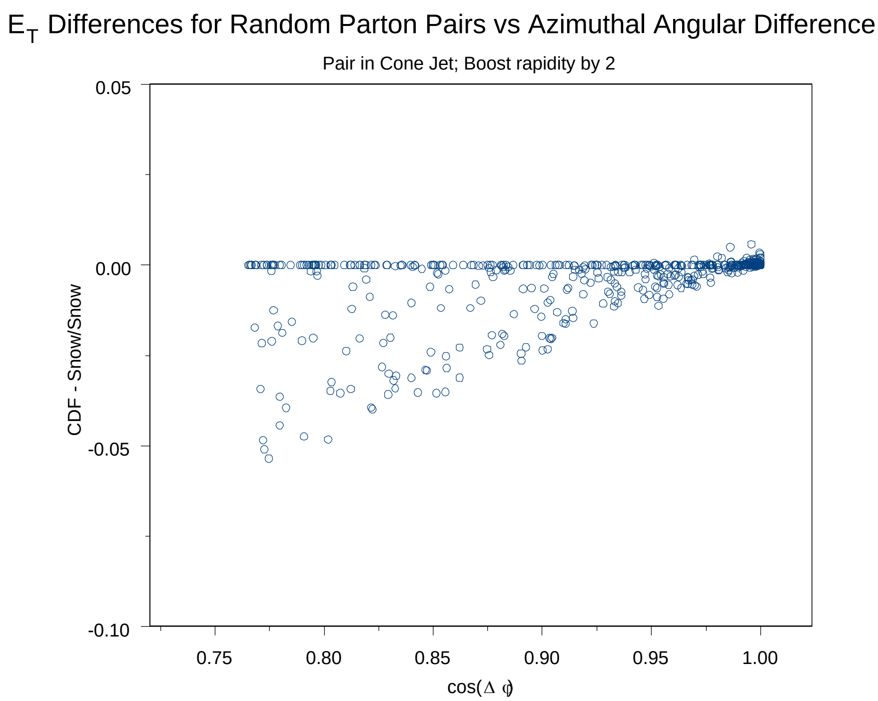
To understand the conversion of cross section ratio to an E_T ratio we need the local log/log derivative of the jet cross section. This is displayed in the next figure for the 4-D case

where the log/log derivative is in the range -6 to -6.5 for E_T in the range of 100 to 200 GeV. At larger E_T values the cross section falls off more rapidly. With the approximate form

$$\frac{d\sigma}{dE_T}(E_T) \simeq \left.\frac{d\sigma}{dE_T}\right|_0 \left(\frac{E_0}{E_T}\right)^n$$

we see that a small displacement in E_T, $E_T \to E_T(1+\delta)$, yields the following (local) fractional change in the

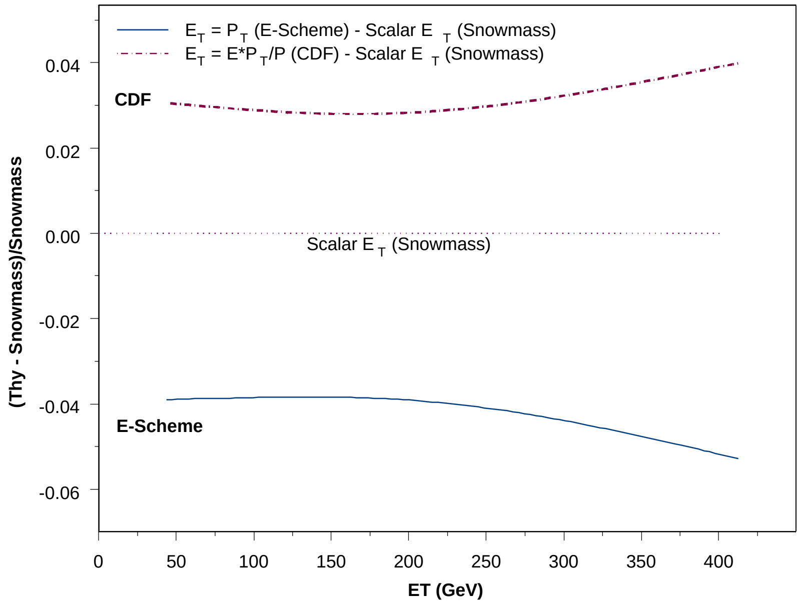


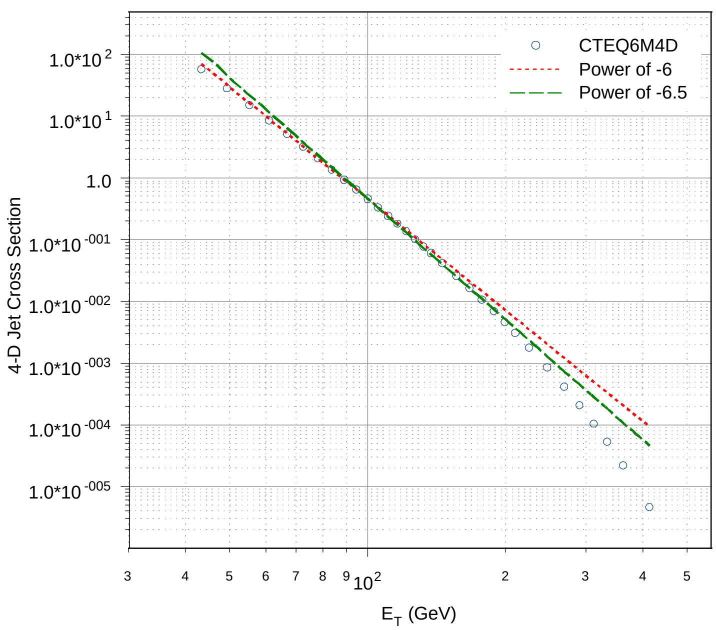


cross section

$$\frac{\frac{d\sigma}{dE_T}\left(E_T\left(1+\delta\right)\right)-\frac{d\sigma}{dE_T}\left(E_T\right)}{\frac{d\sigma}{dE_T}\left(E_T\right)} \simeq (1+\delta)^{-n}-1 \simeq -n\delta.$$

Hence the E_T in the 4-D case is expected to be smaller by about 0.7% in order to yield a 4% decrease in the cross section..

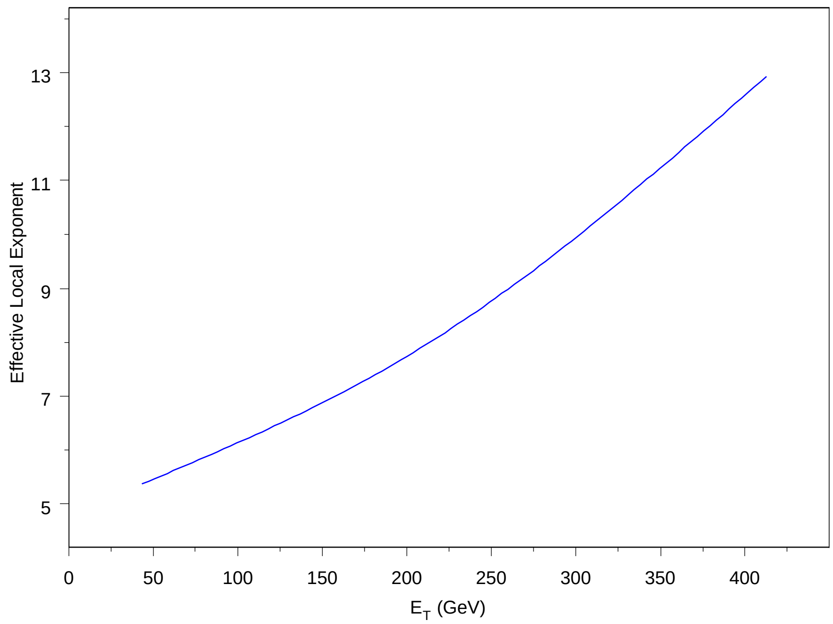




Next consider the difference between the 4-D case and the CDF definition of the jet E_T,

$$R_{E_T\mathrm{CDF}} = \frac{\sqrt{|\overrightarrow{p_J}|^2 + M_J^2}}{|\overrightarrow{p_J}|} = \sqrt{1 + \frac{M_J^2}{|\overrightarrow{p_J}|}} \equiv \sqrt{1 + r_M}.$$

With this last definition we can convert the ratio of the 4-D jet cross sections into an effective mass, scaled by the total momentum. Using the above expressions ($\delta \rightarrow 0.5 r_M$) for the fractional difference between the 2 jet



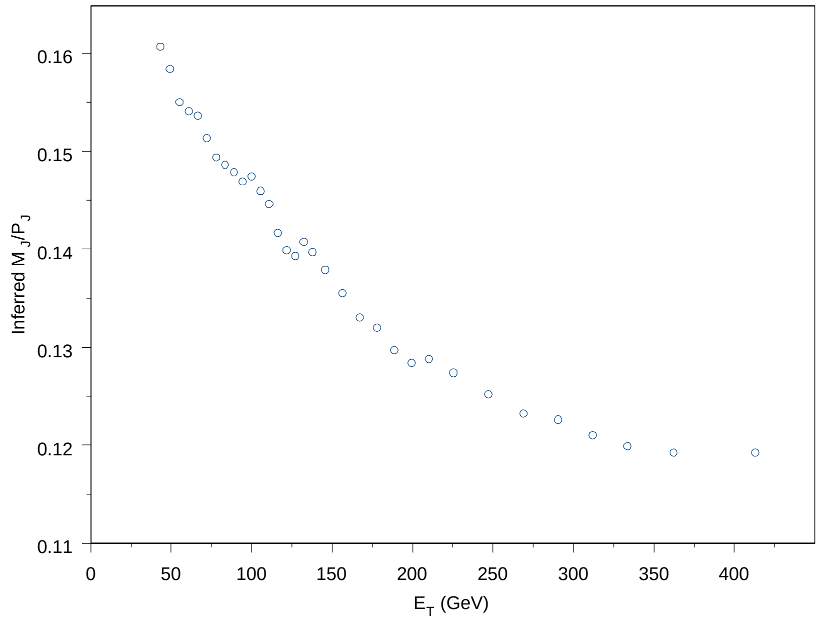
definitions, we obtain the approximate expressions

$$\frac{\sigma_{4D} - \sigma_{4D,CDF}}{\sigma_{4D,CDF}} \simeq -\frac{n}{2} r_M,$$

$$\frac{M_J}{P_J} \simeq \sqrt{\frac{-2}{n}\left(\frac{\sigma_{4D} - \sigma_{4D,CDF}}{\sigma_{4D,CDF}}\right)}.$$

To use this expression over the full range of E_T, we should evaluate the "local" power n as indicated in the next figure.

If we employ this local value for n, we find the following plot. We see that the inferred jet mass is 12% to 16% of the total jet momentum with the smaller values at the larger E_T values, as expected.


Inferred M_J/P_J
0.16
0.15
0.14
0.13
0.12
0.11
0
50
100
150
200
250
300
350
400
E_T (GeV)