(Perturbative) QCD, Jets & Colliders

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Lecture 3: Calculating with QCD –
Hadrons in the Initial State and PDFs
(Correcting the Parton Model)

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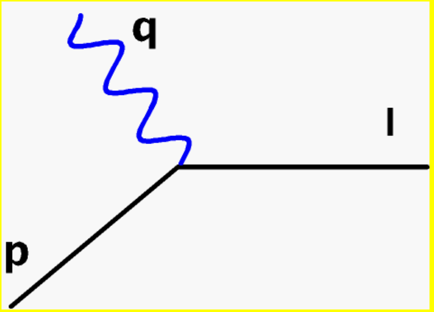
Outline

1. Introduction & (Pre) History – The Parton Model
2. pQCD - e+e- Physics and Perturbation Theory (Correcting the Parton Model)
3. **pQCD - Hadrons in the Initial State and PDFs**
4. pQCD - Hadrons and Jets in the Final State
5. Jets at Work

pQCD Calculation IV -

Parton Distribution Functions in QCD

- Revisit DIS – include real gluon emission (massless partons)



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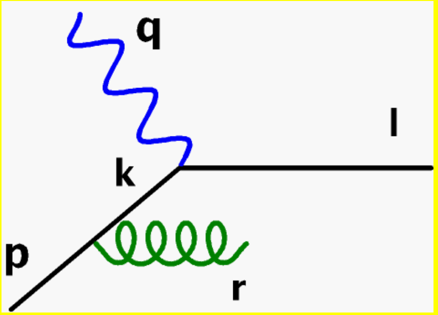
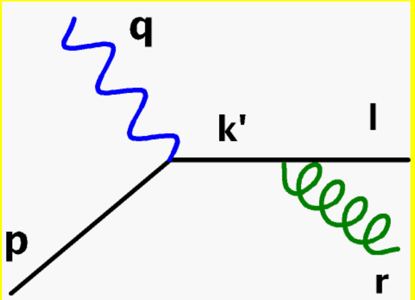


NLO Real



NLO Real

- Singularities arise when the internal propagators go on-shell (collinear and soft gluon emission).

- In the appropriate (light-cone) gauge, the divergent contribution in the middle diagram can be written*

$$\hat{F}_2\Big|_{Div} = e_q^2 \frac{\alpha_s}{2\pi} x\hat{P}_{qq}(x) \int_0^{2\nu} \frac{d\left|k^2\right|}{\left|k^2\right|}$$

* This is gauge dependent - only the sum of the middle and right graphs squared is gauge invariant. In the light-cone gauge only the middle graph is singular.

Singular Configurations -

- the $|k^2|$ integral goes all the way up to the kinematic boundary – it is not cutoff at a fixed (small) value as assumed by the parton model (so expect some differences)

- the $|k^2|$ integral is singular at the lower limit – control with a cutoff κ^2 for now (this "long distance" behavior is non-perturbatively controlled by "confinement" in real life)

- the (collinear) singularity is multiplied by a characteristic function of the quark's momentum fraction *x* – the "splitting function" – that tells us how the longitudinal momentum is shared

$$\hat{P}_{qq}(x) = C_F \frac{1+x^2}{1-x}$$

More on Singular Configurations -

- Put in cutoff and include *all* diagrams above (in standard from) to define for DIS from a quark

$$\hat{F}_{2,q}^{NLO}\left(x,Q^2\right)=e_q^2 x\left[\delta\left(1-x\right)+\frac{\alpha_s}{2\pi}\left(\hat{P}_{qq}\left(x\right)\ln\left(\frac{Q^2}{\kappa^2}\right)+C\left(x\right)\right)\right]$$

where both the collinear term *P*(x) and the non-collinear singular bit *C*(x) are calculable functions in pQCD (i.e., IRS quantities).

⇒ **Conclude! : Scaling is broken** (i.e., the Parton Model) by ln(Q) terms (and we must sum them)!

⇒ The distribution of quarks (in a quark) is now (being explicit about the renormalization scale μ)

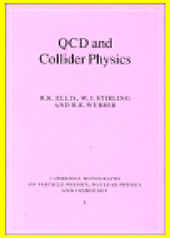
$$q_q\left(x,\frac{Q^2}{\kappa^2},\mu^2\right)=\delta\left(1-x\right)+\frac{\alpha_s\left(\mu^2\right)}{2\pi}\left(\hat{P}_{qq}\left(x\right)\ln\left(\frac{Q^2}{\kappa^2}\right)+C\left(x\right)\right)$$

and quarks are (likely) accompanied by (approximately) collinear gluons

ASIDE: Some calculational details -

- chose the following vectors for the incident quark, light-like gauge fixing vector and virtual photon –

see Chapter 4 in



$$p^{\mu} = \left(P,0,0,P\right); n^{\mu} = \left(\frac{1}{2P},0,0,\frac{-1}{2P}\right); q^{\mu} = \nu n^{\mu} + q_T^{\mu}$$

- such that $p^2 = n^2 = q_T \cdot n = q_T \cdot p = 0 ; n \cdot p = 1 ; q \cdot p = \nu$

$$q_T^2 = -q^2 = Q^2 \, ; \, x = Q^2 \big/ 2\nu$$

- If the emitted gluon has momentum and polarization , we require (conserved current and gauge choice) $\varepsilon \cdot r = \varepsilon \cdot n = 0$

- The momentum of the internal quark leg can be written in terms of a transverse vector k_T (similar to q_T)

$$k^{\mu} = \xi p^{\mu} + \frac{k_T^2 - \left|k^2\right|}{2\xi} n^{\mu} + k_T^{\mu} \, ; \, d^4 k = \frac{d\xi}{2\xi} dk^2 d^2 k_T$$

More details -

- The appropriately summed, averaged (spin and color) and projected matrix element is

$$\frac{1}{4\pi} n^{\alpha} n^{\beta} \overline{\Sigma} \left| M \right|^{2}_{\alpha\beta} = \frac{8 e_q^2 \alpha_s}{\left| k^2 \right|} \xi P(\xi) \quad \text{with } P(\xi = x) \text{ above!}$$

- The 2-body phase space in these variables is

$$d\Phi_2 = \frac{1}{16 \nu \pi^2} \int d\xi \, dk^2 dk_T^2 d\theta \, \delta\left(k_T^2 - (1-\xi)\left|k^2\right|\right) \delta\left(\xi - x - \frac{\left|k^2\right| + 2 q_T \cdot k_T}{2\nu}\right)$$

- Performing all the integrals ($0<\theta<\pi$) except $d\xi$ yields the result above (the δ fcts put the outgoing q and g on-shell).

Include virtual graphs -

- ~$\delta((p+q)^2)$ - Contribute for x=1, $\rightarrow \delta(1\text{-}x)$ term + …
- Quark (baryon) number is conserved*, independent of Q^2

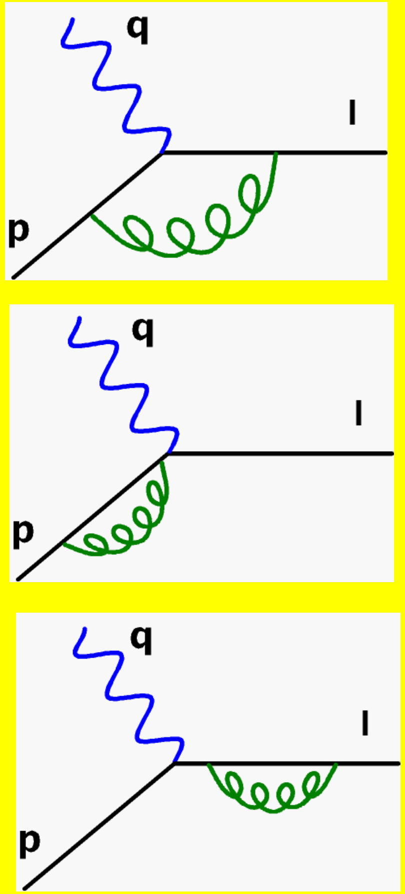
$$\hat{P}_{qq}(x) \to P_{qq}(x) = C_F\left[\frac{1+x^2}{(1-x)_+} + \frac{3}{2}\delta(1-x)\right] = C_F\left(\frac{1+x^2}{1-x}\right)_+$$

where the "+" distribution is defined by

$$\int_0^1 dx\, f(x)\left(\frac{1+x^2}{1-x}\right)_+ \equiv \int_0^1 dx\left[f(x) - f(1)\right]\left(\frac{1+x^2}{1-x}\right)$$

- With care taken below for the process $g \to q\bar{q}$, this is just (i.e., due to the delta fct, virtual bit)

$$\hat{P}(x) \to \hat{P}(x)_+ = P(x)$$



*Confirm quark number conservation - HW

Put it together -

- For a quark in a proton, as an intermediate step we introduce a "bare" quark distribution q_0 and convolute with above

$$q\left(x,\frac{Q^2}{\kappa^2},\mu^2\right)=q_0(x)+\frac{\alpha_s(\mu^2)}{2\pi}\int_x^1\frac{d\xi}{\xi}q_0(\xi)\left[P_{qq}\left(\frac{x}{\xi}\right)\ln\left(\frac{Q^2}{\kappa^2}\right)+C_q\left(\frac{x}{\xi}\right)\right]+\ldots$$

- q_0 plays similar role to $\alpha_s(M)$ used earlier – an "unphysical" place to hide infinities. The theory is well behaved but our approach in terms of "bare" objects requires us to follow this round-about path.

- Need to get rid of the "cut-off" κ and the "bare" distribution

Factorization Scale -

Introduce a *factorization* scale μ_F – "absorb" collinear singularities (for $|k^2|<\mu_F^2$) into the bare distribution and obtain the regularized, scale dependent distribution, *i.e.*, the long distance physics is all in the regularized distribution.

- Define $$\ln\left(\frac{Q^2}{\kappa^2}\right) = \ln\left(\frac{Q^2}{\mu_F^2}\right) + \ln\left(\frac{\mu_F^2}{\kappa^2}\right)$$
- Split the non-collinear term in a factorization scheme dependent fashion where the second term will be included in the long distance physics (an arbitrary choice) $$C_q(z) \equiv \bar{C}_q(z) + \tilde{C}_q(z)$$
- Physical quantities are scheme independent and the calculation will be also if <u>all</u> parts are performed in the same scheme!
- E.g., the DIS choice is to absorb everything, $\bar{C}_q^{DIS} = 0$

Factorization -

- Finally, choosing the factorization scale to equal the renormalization scale (simplifying but not necessary), $\mu_F^2 = \mu^2$, define

$$q_q\left(x,\mu^2\right) = q_0\left(x\right) + \frac{\alpha_s\left(\mu^2\right)}{2\pi}\int_x^1 \frac{d\xi}{\xi} q_0\left(\xi\right)\left[P_{qq}\left(\frac{x}{\xi}\right)\ln\left(\frac{\mu^2}{\kappa^2}\right) + \tilde{C}_q\left(\frac{x}{\xi}\right)\right] + \ldots$$

which formally includes all of the collinear structure, and is thus not calculable in pQCD, but allows us to write

$$q_q\left(x,Q^2\right) = \int_x^1 \frac{d\xi}{\xi} q_q\left(\frac{x}{\xi},\mu^2\right)\left[\delta\left(1-\frac{x}{\xi}\right) + \frac{\alpha_s}{2\pi}\left\{P_{qq}\left(\frac{x}{\xi}\right)\ln\left(\frac{Q^2}{\mu^2}\right) + \bar{C}_q\left(\frac{x}{\xi}\right)\right\} + \ldots\right]$$

$$F_{2,q}^{NLO}\left(x,Q^2\right) = x\sum_{q,\bar{q}} e_q^2 \int_x^1 \frac{d\xi}{\xi} \; q_q\left(\xi,\mu^2\right)$$

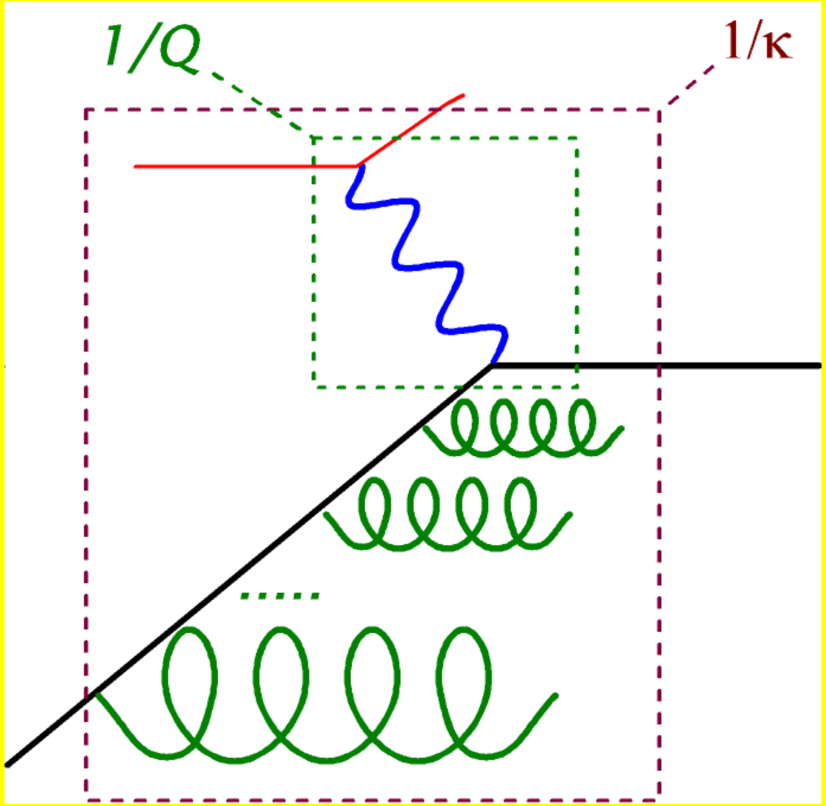
$$\times\left[\delta\left(1-\frac{x}{\xi}\right) + \frac{\alpha_s\left(\mu^2\right)}{2\pi}\left\{P_{qq}\left(\frac{x}{\xi}\right)\ln\left(\frac{Q^2}{\mu^2}\right) + \bar{C}_q\left(\frac{x}{\xi}\right)\right\} + \ldots\right]$$

Summary in pictures, first with cut-off

- Order-by-order, we are summing the largest contributions of the emission of multiple gluons
- The change in size (wavelength) of the gluons represents the strong ordering of the transverse momenta (smaller wavelength means larger momentum)

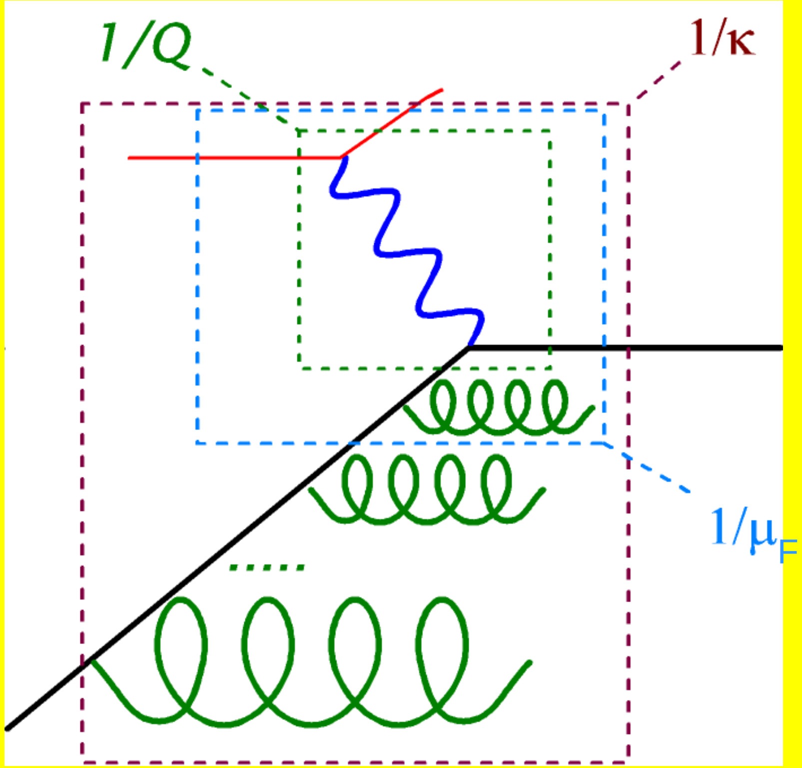
$$k_{T,1}^2 << k_{T,2}^2 << \cdots << k_{T,n}^2$$

With cut-off κ



Summary in pictures, with factorization scale

- Separate contributions above and below the factorization scale μ
- And factor scales κ to μ into the renormalized distribution – leaving



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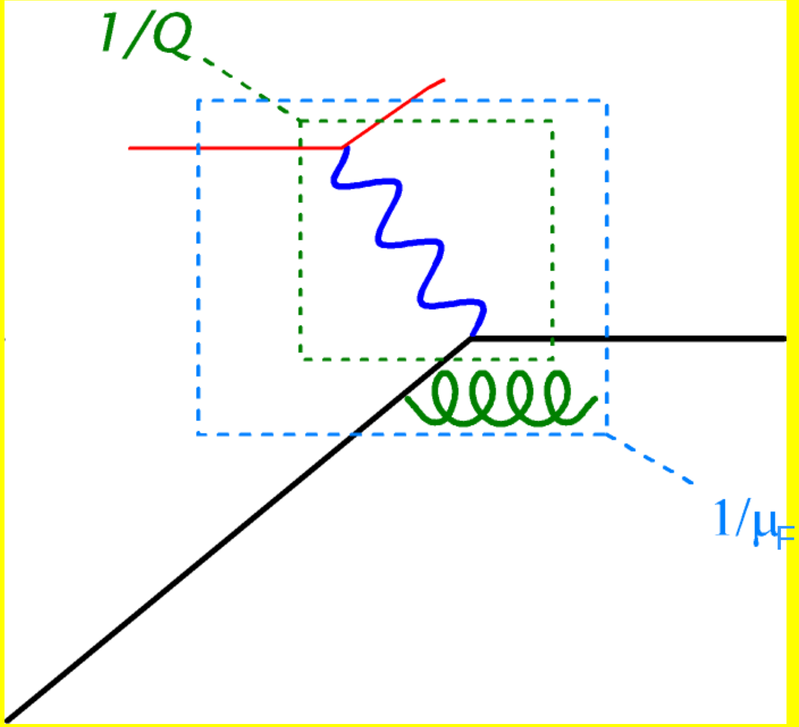
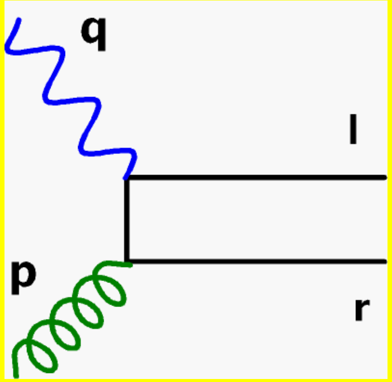


One more addition -

- At this order we also have (a quark from a gluon) yielding



$$\hat{F}_{2,g}^{NLO}\left(x,Q^2\right)=x\sum_{q,\bar{q}}e_q^2\frac{\alpha_s}{2\pi}\left[P_{qg}\left(x\right)\ln\left(\frac{Q^2}{\kappa^2}\right)+C_g\left(x\right)\right]$$

$$P_{qg}\left(x\right)=T_R\left[x^2+\left(1-x\right)^2\right];\quad T_R=\frac{1}{2}$$

- So we really want a "bare" gluon distribution too -

$$q\left(x,\mu^2\right)=q_0\left(x\right)+\frac{\alpha_s}{2\pi}\int_x^1\frac{d\xi}{\xi}q_0\left(\xi\right)\left[P_{qq}\left(\frac{x}{\xi}\right)\ln\left(\frac{\mu^2}{\kappa^2}\right)+\tilde{C}_q\left(\frac{x}{\xi}\right)\right]$$

$$+\frac{\alpha_s}{2\pi}\int_x^1\frac{d\xi}{\xi}g_0\left(\xi\right)\left[P_{qg}\left(\frac{x}{\xi}\right)\ln\left(\frac{\mu^2}{\kappa^2}\right)+\tilde{C}_g\left(\frac{x}{\xi}\right)\right]+\ldots$$

Include glue and …

- Factorizing with the choice Q = μ so only 1 scale

$$q\left(x,\mu^2\right)=\int_x^1\frac{d\xi}{\xi}q\left(\xi,\mu^2\right)\left[\delta\left(1-\frac{x}{\xi}\right)+\frac{\alpha_s\left(\mu^2\right)}{2\pi}\bar{C}_q^{\overline{MS}}\left(\frac{x}{\xi}\right)+\ldots\right]+$$

$$+\int_x^1\frac{d\xi}{\xi}g\left(\xi,\mu^2\right)\left[\frac{\alpha_s\left(\mu^2\right)}{2\pi}\bar{C}_q^{\overline{MS}}\left(\frac{x}{\xi}\right)+\ldots\right]$$

- Recall the specific form of the “finite” piece, C(x) [called the *coefficient function*], depends on the renormalization scheme and on the specific quantity being calculated [*e*.*g*., different for F_1 and F_2].

Coefficient Functions

$$\bar{C}_q^{\overline{MS}}(z) = C_F\left[2\left(\frac{\ln(1-z)}{1-z}\right)_+ - \frac{3}{2}\left(\frac{1}{1-z}\right)_+ - (1+z)\ln(1-z)\right.$$

$$\left. - \frac{1+z^2}{1-z}\ln z + 3 + 2z - \left(\frac{\pi^2}{3} + \frac{9}{2}\right)\delta(1-z)\right],$$

$$\bar{C}_g^{\overline{MS}}(z) = T_R\left[\left((1-z)^2 + z^2\right)\left(\frac{\ln(1-z)}{z}\right) - 8z^2 + 8z - 1\right].$$

Finally for DIS proton in $\overline{MS}(\mu = Q)$

$$F_2\left(x,Q^2\right) = x\sum_{q,\bar{q}} e_q^2 \int_x^1 \frac{d\xi}{\xi} q\left(\xi,Q^2\right)\left[\delta\left(1-\frac{x}{\xi}\right) + \frac{\alpha_s\left(Q^2\right)}{2\pi} C_q^{\overline{MS}}\left(\frac{x}{\xi}\right) + \ldots\right] +$$

$$+ x\sum_{q,\bar{q}} e_q^2 \int_x^1 \frac{d\xi}{\xi} g\left(\xi,Q^2\right)\left[\frac{\alpha_s\left(Q^2\right)}{2\pi} C_q^{\overline{MS}}\left(\frac{x}{\xi}\right) + \ldots\right]$$

- As with the renormalized, running coupling, pQCD does *not* tell us about the full, running parton distributions. These must be determined experimentally.

- pQCD does tell us how they *evolve* with the scale μ

General Structure of Convolution -

- Consider the general version ($\mu^2 \neq \mu_F^2 \neq Q^2$)

$$F_2^{\gamma N}\left(x,Q^2\right) = x\sum_{q,\bar{q}} e_q^2 \int_x^1 \frac{d\xi}{\xi} q^{\overline{MS}}\left(\xi,\mu_F^2,\alpha_s\left(\mu\right)\right)$$

$$\times\left[\delta\left(1-\frac{x}{\xi}\right)+\frac{\alpha_s\left(\mu\right)}{2\pi}\left\{P_{qq}\left(\frac{x}{\xi}\right)\ln\left(\frac{Q^2}{\mu_F^2}\right)+C_q^{\overline{MS}}\left(\frac{x}{\xi}\right)\right\}+\ldots\right]$$

$$+x\sum_{q,\bar{q}} e_q^2 \int_x^1 \frac{d\xi}{\xi} g^{\overline{MS}}\left(\xi,\mu_F^2,\alpha_s\left(\mu\right)\right) \left[\frac{\alpha_s\left(\mu\right)}{2\pi}\left\{P_{qg}\left(\frac{x}{\xi}\right)\ln\left(\frac{Q^2}{\mu_F^2}\right)+C_g^{\overline{MS}}\left(\frac{x}{\xi}\right)\right\}+\ldots\right]$$

$$\equiv \sum_{a=q,\bar{q},g} C_2^{\gamma a}\left(\frac{x}{\xi},\frac{Q}{\mu_F},\frac{\mu_F}{\mu},\alpha_s\left(\mu\right)\right)\otimes f_{a/N}\left(\xi,\mu_F^2,\alpha_s\left(\mu\right)\right)$$

- Convolution of (non-perturbative) long distance physics with short-distance (perturbative) IRS physics (defined by factorization scale) – μ^2 and μ_F^2 dependence must be matched between the 2 components - **The General Structure of pQCD** -

DGLAP

- Consider the general version ($\mu^2 = \mu_F^2 \neq Q^2$)

$$F_2\left(x,Q^2\right) = x\sum_{q,\bar{q}} e_q^2 \int_x^1 \frac{d\xi}{\xi} q^{\overline{MS}}\left(\xi,\mu^2\right)$$
$$\times\left[\delta\left(1-\frac{x}{\xi}\right)+\frac{\alpha_s\left(\mu\right)}{2\pi}\left\{P_{qq}\left(\frac{x}{\xi}\right)\ln\left(\frac{Q^2}{\mu^2}\right)+\bar{C}_q^{\overline{MS}}\left(\frac{x}{\xi}\right)\right\}+\ldots\right]$$
$$+x\sum_{q,\bar{q}} e_q^2 \int_x^1 \frac{d\xi}{\xi} g^{\overline{MS}}\left(\xi,\mu^2\right)\left[\frac{\alpha_s\left(\mu\right)}{2\pi}\left\{P_{qg}\left(\frac{x}{\xi}\right)\ln\left(\frac{Q^2}{\mu^2}\right)+\bar{C}_g^{\overline{MS}}\left(\frac{x}{\xi}\right)\right\}+\ldots\right]$$

LHS is μ independent so RHS must be also, order-by-order in pQCD
$\Rightarrow$ DGLAP – (Dokshitzer-Gribov-Lipatov-Altarelli-Parisi)

DGLAP – Perturbative condition on NONperturbative quantity

$$\mu^2 \frac{\partial}{\partial \mu^2} q\left(x,\mu^2\right) = \frac{\alpha_s\left(\mu^2\right)}{2\pi} \int_x^1 \frac{d\xi}{\xi} P\left(\frac{x}{\xi}\right) q\left(\xi,\mu^2\right)$$

$$= \frac{\alpha_s\left(\mu^2\right)}{2\pi} \int_x^1 \frac{dz}{z} P\left(z\right) q\left(\frac{x}{z},\mu^2\right)$$

- The splitting function *P* (like the β function) is what is calculable in pQCD.

$$P\left(z,\alpha_s\left(\mu^2\right)\right) = P^{(0)}(z) + \frac{\alpha_s\left(\mu^2\right)}{2\pi} P^{(1)}\left(z\right) + \ldots$$

- The splitting function $P_{ab}\left(z\right)$ can be interpreted as the *probability* to find a parton of type *a* in a parton of type *b* with a fraction *z* of its longitudinal momentum and transverse momentum < μ, per unit log k_T

DGLAP -

- We really have a matrix problem ($2n_f+1$ dimensional)

$$\mu^2 \frac{\partial}{\partial \mu^2} \begin{pmatrix} q_i\left(x,\mu^2\right) \\ g\left(x,\mu^2\right) \end{pmatrix} = \frac{\alpha_s\left(\mu^2\right)}{2\pi} \sum_{q_j,\bar{q}_j} \int_x^1 \frac{d\xi}{\xi} \times$$

$$\begin{pmatrix} P_{q_i q_j}\left(\frac{x}{\xi},\alpha_s\left(\mu^2\right)\right) & P_{q_i g}\left(\frac{x}{\xi},\alpha_s\left(\mu^2\right)\right) \\ P_{g q_j}\left(\frac{x}{\xi},\alpha_s\left(\mu^2\right)\right) & P_{gg}\left(\frac{x}{\xi},\alpha_s\left(\mu^2\right)\right) \end{pmatrix} \begin{pmatrix} q_j\left(\xi,\mu^2\right) \\ g\left(\xi,\mu^2\right) \end{pmatrix}$$

- Luckily, symmetries come to our rescue (charge conjugation, SU(n_f),…) –

$$P_{q_i q_j} = P_{\bar{q}_i \bar{q}_j} \quad P_{q_i \bar{q}_j} = P_{\bar{q}_i q_j} \quad P_{q_i g} = P_{\bar{q}_i g} \equiv P_{qg} \quad P_{g q_i} = P_{g \bar{q}_i} \equiv P_{gq}$$

- QCD is flavor blind and, at leading order, is flavor diagonal

So …

- Quark number and momentum conservation means***

$$\int_0^1 dx P_{qq}^{(0)}(x) = 0 \quad \int_0^1 dx\, x\left[P_{qq}^{(0)}(x) + P_{gq}^{(0)}(x)\right] = 0 \quad \int_0^1 dx\, x\left[2n_f P_{qg}^{(0)}(x) + P_{gg}^{(0)}(x)\right] = 0$$

- In summary

$$P_{qq}^{(0)}(x) = C_F\left[\frac{1+x^2}{(1-x)_+} + \frac{3}{2}\delta(1-x)\right] = C_F\left(\frac{1+x^2}{1-x}\right)_+$$

$$P_{qg}^{(0)}(x) = T_R\left[x^2 + (1-x)^2\right]$$

$$P_{gq}^{(0)}(x) = C_F\left[\frac{1+(1-x)^2}{x}\right]$$

$$P_{gg}^{(0)}(x) = 2C_A\left[\frac{x}{(1-x)_+} + \frac{(1-x)}{x} + x(1-x)\right] + \delta(1-x)\frac{11C_A - 4n_f T_R}{6}$$

*** Verify these sum rules - see HW

DGLAP & Moments -

- We can explore the DGLAP equation by taking moments – define

$$f\left(j,\mu^2\right)=\int_0^1 dx\, x^{j-1} f\left(x,\mu^2\right),\ f=q_i,\, g$$

- With inverse (contour *C* parallel to the imaginary axis and to the right of all singularities)

$$f\left(x,\mu^2\right)\equiv\frac{1}{2\pi i}\int_C dj\ x^{-j} f\left(j,\mu^2\right)$$

- For a non-singlet quark distribution, $q_{NS}=q-\overline{q}$, with evolution controlled by P_{qq}

$$\mu^2\frac{\partial}{\partial\mu^2}q_{NS}\left(j,\mu^2\right)=\frac{\alpha_s\left(\mu^2\right)}{2\pi}\gamma_{qq}\left(j,\alpha_s\left(\mu^2\right)\right)q_{NS}\left(j,\mu^2\right)$$

$$\gamma_{qq}\left(j,\alpha_s\left(\mu^2\right)\right)=\int_0^1 dx\, x^{j-1}P_{qq}\left(x,\alpha_s\left(\mu^2\right)\right)$$

Anomalous dimensions -

- In leading order (1-loop – no μ dependence in *P*) the solution is

$$q_{NS}\left(j,\mu^2\right)=\left(\frac{\ln\left(\mu^2/\Lambda_{QCD}^2\right)}{\ln\left(\mu_0^2/\Lambda_{QCD}^2\right)}\right)^{\frac{2\gamma_{qq}^{(0)}(j)}{\beta_0}} q_{NS}\left(j,\mu_0^2\right)$$

This behavior is characteristic of gauge theories where $\gamma_{qq}^{(0)}(j)$ is often called the "anomalous dimension" for the j[th] moment.

- ASIDE: If this were a theory with a fixed (not running) coupling, $\alpha_s(\mu)\to\alpha_0$ we would find

$$q_0\left(j,\mu^2\right)=q_0\left(j,\mu_0^2\right)\left(\frac{\mu^2}{\mu_0^2}\right)^{\alpha_0\gamma_{qq}(j)/2\pi}$$

which makes the label anomalous dimension more clear. In such a theory the evolution is very fast and hard partons are very unlikely!

Singlet Distribution -

- In a similar way we can study the moments of the singlet distribution $\Sigma\left(x,\mu^2\right) \equiv \sum_i \left[q_i\left(x,\mu^2\right) + \bar{q}_i\left(x,\mu^2\right) \right]$,which mixes with the gluon
- Its moments obey a vector/matrix equation

$$\mu^2 \frac{\partial}{\partial\mu^2} \begin{pmatrix} \Sigma\left(j,\mu^2\right) \\ g\left(j,\mu^2\right) \end{pmatrix} = \frac{\alpha_s\left(\mu^2\right)}{2\pi} \times$$

$$\begin{pmatrix} \gamma_{qq}\left(j,\alpha_s\left(\mu^2\right)\right) & 2n_f \gamma_{qg}\left(j,\alpha_s\left(\mu^2\right)\right) \\ \gamma_{gq}\left(j,\alpha_s\left(\mu^2\right)\right) & \gamma_{gg}\left(j,\alpha_s\left(\mu^2\right)\right) \end{pmatrix} \begin{pmatrix} \Sigma\left(j,\mu^2\right) \\ g\left(j,\mu^2\right) \end{pmatrix}$$

Anomalous dimensions II -

- The explicit (1-loop) anomalous dimensions are***

$$\gamma_{qq}^{(0)}(j) = C_F\left[-\frac{1}{2} + \frac{1}{j(j+1)} - 2\sum_{k=2}^{j}\frac{1}{k}\right]$$

$$\gamma_{qg}^{(0)}(j) = T_R\left[\frac{2+j+j^2}{j(j+1)(j+2)}\right]$$

$$\gamma_{gq}^{(0)}(j) = C_F\left[\frac{2+j+j^2}{j(j^2-1)}\right]$$

$$\gamma_{gg}^{(0)}(j) = 2C_A\left[-\frac{1}{12} + \frac{1}{j(j-1)} + \frac{1}{(j+1)(j+2)} - \sum_{k=2}^{j}\frac{1}{k}\right] - \left(\frac{2}{3}\right)n_f T_R$$

- The moments can be inverted with the inverse Mellin transformation (at least numerically).

*** See HW

Examples/Conclusions -

- Since $d_{qq}(1)=0$, the number of valence quarks does not evolve – flavor is conserved by QCD

- Since $d_{qq}(j \geq 2) < 0$, the non-singlet quark distribution evolves by decreasing at large *x* and increasing at small *x* – *as expected as the quark emits gluons*

- Next note that

$$\int_0^1 dx\, x^{j-1}\frac{1}{x} = \frac{1}{j-1}$$

$$\int dx\, x^{j-1}\frac{1}{(1-x)_+} = -\int_0^1 dx\,\frac{x^{j-1}-1}{x-1} \sim \ln j \quad j >> 1$$

The pole at *j=1* means the fixed order analysis is unreliable for the limit of small x

More -

- The large x behavior can be inferred from the ln *j* behavior of the anomalous dimensions and the fact that

$$f\left(x,\mu^2\right)\xrightarrow[x\to 1]{}\left(1-x\right)^{a\left(\mu^2\right)} \Rightarrow f\left(j,\mu^2\right)\xrightarrow[j>>1]{} j^{-a\left(\mu^2\right)}$$

- Thus we find

$$\gamma_{qq}^{(0)}\left(j>>1\right)\sim -4C_F \ln j \quad \Rightarrow$$

$$j^{-a\left(\mu_0^2\right)}\left(\frac{\ln\left(\mu^2/\Lambda_{QCD}^2\right)}{\ln\left(\mu_0^2/\Lambda_{QCD}^2\right)}\right)^{-4C_F \ln j/\beta_0} = j^{-a\left(\mu_0^2\right)-4C_F \ln\left(\ln\left(\mu^2/\Lambda_{QCD}^2\right)/\ln\left(\mu_0^2/\Lambda_{QCD}^2\right)\right)/\beta_0}$$

$$\Rightarrow q_{NS}\left(x,\mu^2\right)\sim (1-x)^{a\left(\mu_0^2\right)+4C_F \ln\left(\ln\left(\mu^2/\Lambda_{QCD}^2\right)/\ln\left(\mu_0^2/\Lambda_{QCD}^2\right)\right)/\beta_0}$$

Quark Singlet + Gluon System

- For the singlet plus gluon system we must find the corresponding eigenvalues and eigenvectors. With j=2, the momentum sum integral, we have

Eigenvector	Eigenvalue
$\Sigma(2,\mu^2)+g(2,\mu^2)$	0
$\Sigma(2,\mu^2)-(n_f/4C_F)\, g(2,\mu^2)$	$-4C_F/3 - n_f/3$

- The first line confirms that total momentum is conserved during evolution!!

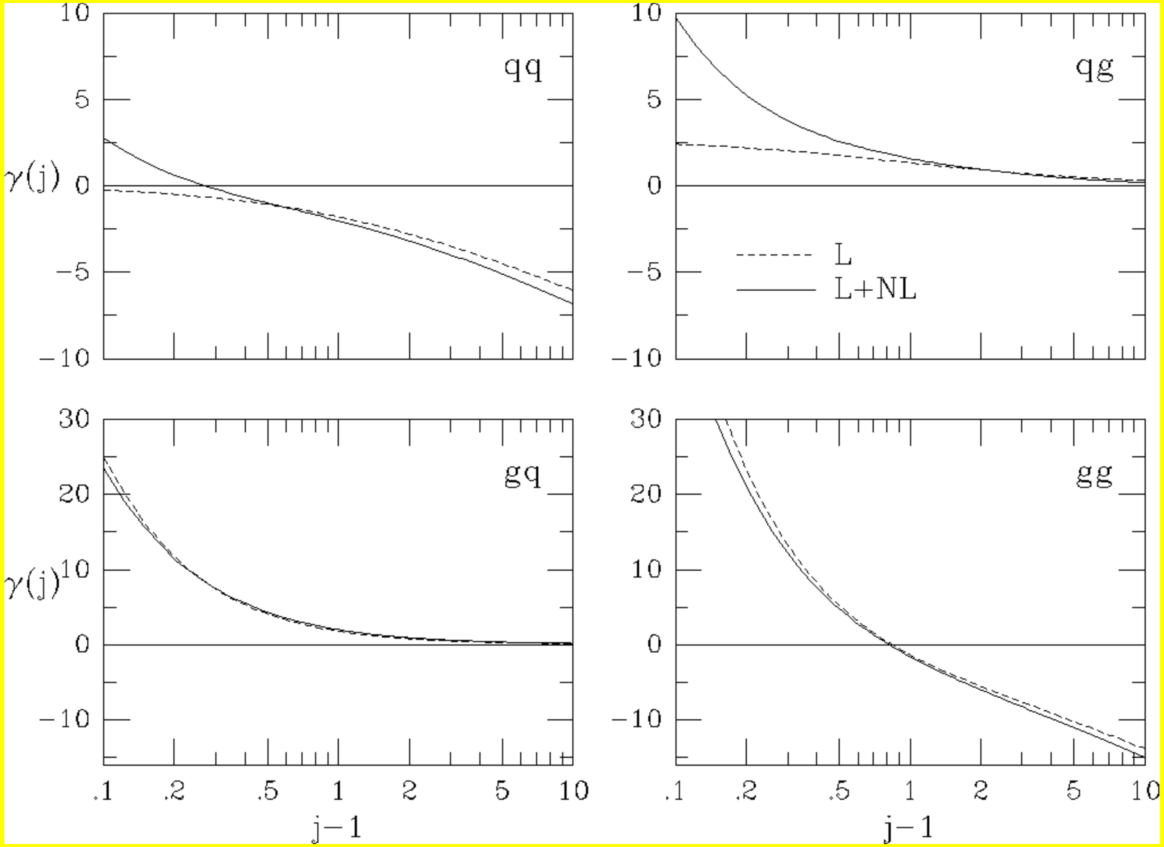
- Since the second eigenvalue is < 0, the second eigenvector vanishes asymptotically (ln $\mu \rightarrow$ infinity)

$$\Sigma\left(2,\mu^2\right)-\frac{n_f}{4C_F}g\left(2,\mu^2\right)\xrightarrow[\ln\mu^2\to\infty]{}0\Rightarrow\frac{\Sigma\left(2,\mu^2\right)}{g\left(2,\mu^2\right)}\xrightarrow[\ln\mu^2\to\infty]{}\frac{n_f}{4C_F}=\frac{3n_f}{16}$$

- Hence the (truly!) asymptotic momentum ratios (at leading order) are

$$f_q = \frac{3n_f}{16+3n_f} \xrightarrow[n_f=6]{} 53\% \qquad f_g = \frac{16}{16+3n_f} \xrightarrow[n_f=6]{} 47\%$$

- Numerically the anomalous dimensions look like –



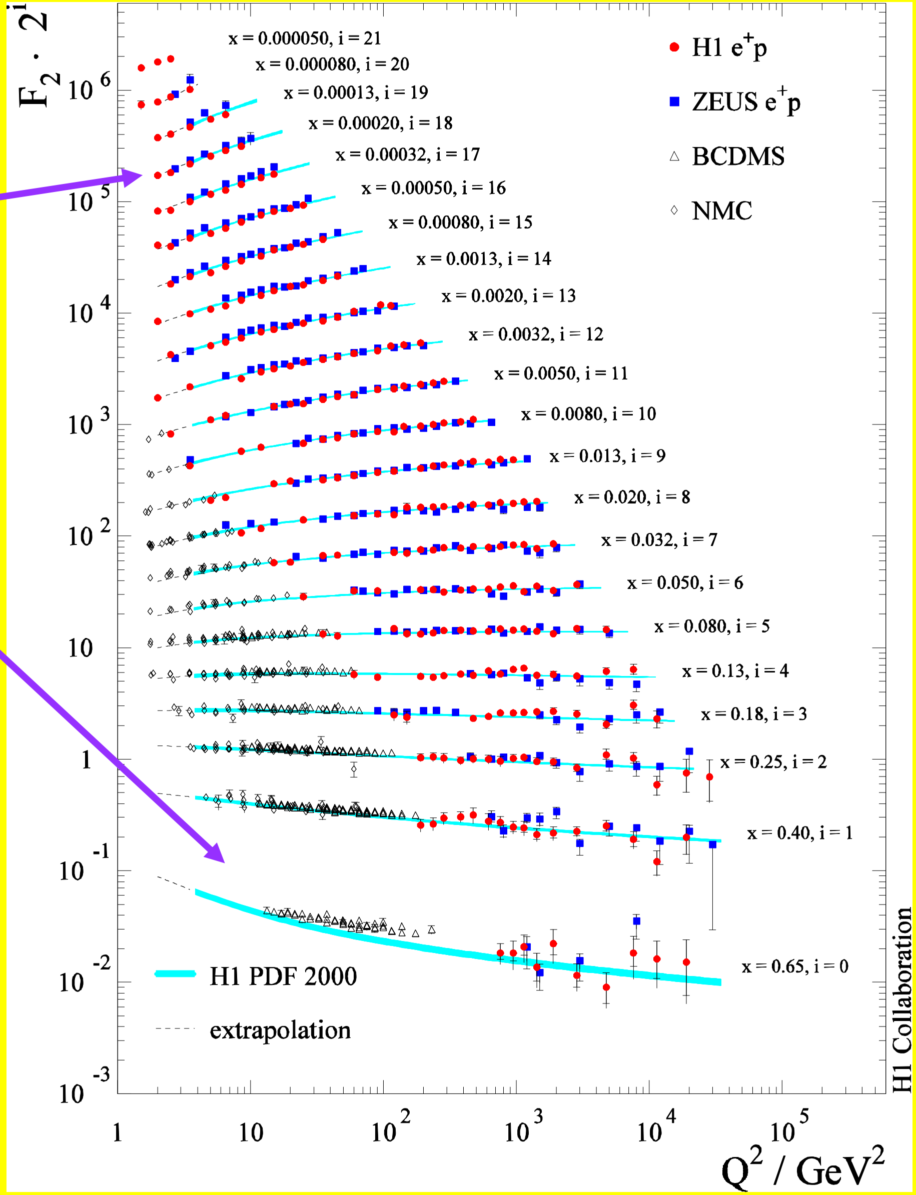
Conclude -

- We expect that the distributions

  increase at small *x*

  decrease at large *x*

  as μ = Q increases, and we see this experimentally.



Global Fits -

- pQCD (as will we see) allows us to describe a broad range of experiments in terms of PDFs

- Determine PDFs from GLOBAL fits to a range of data, now including "propagation" of uncertainties in data with range of fits

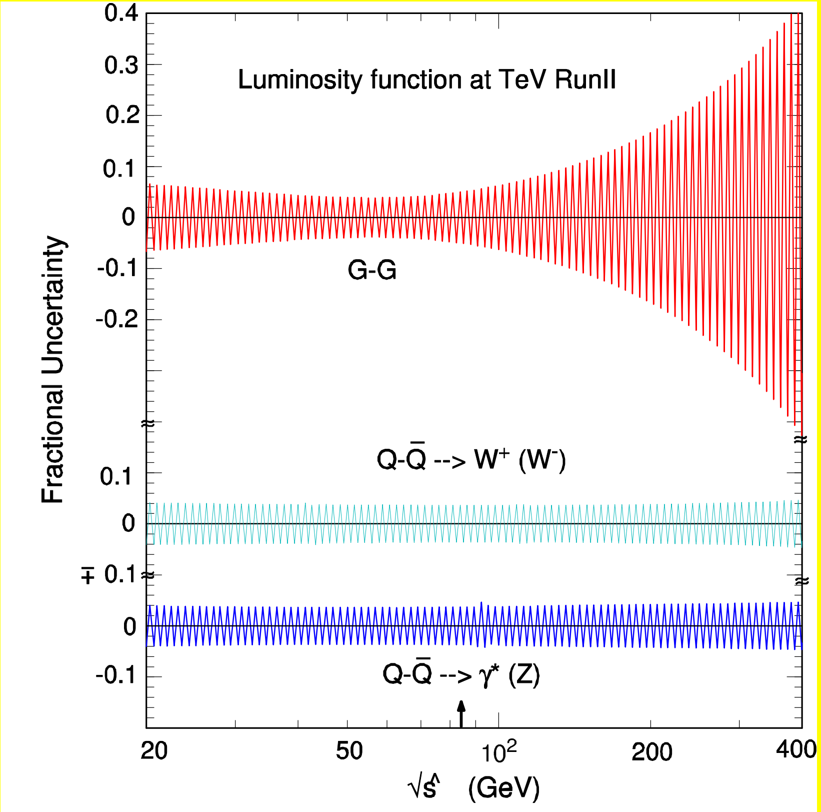
- CTEQ – http://www.phys.psu.edu/~cteq

- MRST - http://durpdg.dur.ac.uk/HEPDATA/HEPDATA.html

- See also - http://hepforge.cedar.ac.uk/

Current Status

CTEQ



$$L = \int dx_1 dx_2 g(x_1,\mu) g(x_2,\mu) \delta\left(x_1 x_2 - \hat{s}/s\right)$$

Where $\sqrt{s}$ is the total hadronic energy, and $\sqrt{\hat{s}}$ is the total partonic, hard scattering energy

$\Rightarrow$ Uncertainties < 10% except for large *x* gluons (just where we need them!)

Measures of parton luminosity uncertainties

Current PDF issues

- More precision for the Gluons
- Flavor, charge asymmetries, e.g., s vs $\bar{s}$
- Heavy flavors (*c*,*b*)
  experimental determination
  include mass effects, defining thresholds
  role of nonperturbative effects (i.e., besides perturbative gluon splitting)
- Do we need NNLO fits? (global data probably not that good yet)

ASIDE: Sudakov Form Factor -

- Consider the function $$\Delta\left(\mu^2,\mu_0^2\right) \equiv \exp\left[-\int_{\mu_0^2}^{\mu^2} \frac{d\kappa^2}{\kappa^2} \int^1 dz \frac{\alpha_s}{2\pi} \hat{P}(z)\right]$$

  which involves the unregulated version of the splitting function but, in a sense, contains the information about the regulation of the soft singularity ($z \to 1$).

- This is the bare* version of the Sudakov Form Factor mentioned earlier.

- Using $P(z) = \hat{P}(z)_+$, and $P(z) = 0$ outside of $0 \le z \le 1$, we can write

$$\mu^2 \frac{\partial}{\partial \mu^2} q\left(x,\mu^2\right) = \frac{\alpha_s\left(\mu^2\right)}{2\pi} \int_x^1 \frac{dz}{z} \hat{P}(z)\, q\left(\frac{x}{z},\mu^2\right) + \frac{q\left(x,\mu^2\right)}{\Delta\left(\mu^2,\mu_0^2\right)} \mu^2 \frac{\partial}{\partial \mu^2} \Delta\left(\mu^2,\mu_0^2\right)$$

* In physical applications the physics will control the soft singularity as was displayed earlier.

In Detail

$$\mu^2 \frac{\partial}{\partial \mu^2} q\left(x,\mu^2\right) = \frac{\alpha_s\left(\mu^2\right)}{2\pi} \int_x^1 \frac{dz}{z} P(z) q\left(\frac{x}{z},\mu^2\right) = \frac{\alpha_s\left(\mu^2\right)}{2\pi} \int_x^1 \frac{dz}{z} \hat{P}(z)_+ q\left(\frac{x}{z},\mu^2\right)$$

$$= \frac{\alpha_s\left(\mu^2\right)}{2\pi} \int_x^1 \frac{dz}{z} \hat{P}(z) q\left(\frac{x}{z},\mu^2\right) - \left[\frac{\alpha_s\left(\mu^2\right)}{2\pi z} q\left(\frac{x}{z},\mu^2\right)\right]_{z=1} \int_0^1 dz \hat{P}(z)$$

$$= \frac{\alpha_s\left(\mu^2\right)}{2\pi} \int_x^1 \frac{dz}{z} \hat{P}(z) q\left(\frac{x}{z},\mu^2\right) - \frac{\alpha_s\left(\mu^2\right)}{2\pi} q\left(x,\mu^2\right) \int_0^1 dz \hat{P}(z)$$

$$= \frac{\alpha_s\left(\mu^2\right)}{2\pi} \int_x^1 \frac{dz}{z} \hat{P}(z) q\left(\frac{x}{z},\mu^2\right) + \frac{q\left(x,\mu^2\right)}{\Delta\left(\mu^2,\mu_0^2\right)} \mu^2 \frac{\partial}{\partial \mu^2} \Delta\left(\mu^2,\mu_0^2\right)$$

Cont'd

- Or, more compactly,

$$\mu^2 \frac{\partial}{\partial \mu^2}\left(\frac{q\left(x,\mu^2\right)}{\Delta\left(\mu^2,\mu_0^2\right)}\right) = \frac{1}{\Delta\left(\mu^2,\mu_0^2\right)}\frac{\alpha_s\left(\mu^2\right)}{2\pi}\int_x^1 \frac{dz}{z}\hat{P}(z)\, q\left(\frac{x}{z},\mu^2\right)$$

- The solution can be written

$$q\left(x,\mu^2\right) = \Delta\left(\mu^2,\mu_0^2\right) q\left(x,\mu_0^2\right) + \int_{\mu_0^2}^{\mu^2} \frac{d\kappa^2}{\kappa^2}\frac{\Delta\left(\mu^2,\mu_0^2\right)}{\Delta\left(\kappa^2,\mu_0^2\right)}\frac{\alpha_s\left(\kappa^2\right)}{2\pi}\int_x^1 \frac{dz}{z}\hat{P}(z)\, q\left(\frac{x}{z},\kappa^2\right)$$

- So we interpret

$\Delta\left(\mu^2,\mu_0^2\right)$ as the probability to evolve without splitting $\mu_0^2 \to \mu^2$

$\Delta\left(\mu^2,\mu_0^2\right)/\Delta\left(\kappa^2,\mu_0^2\right)$ as the probability to evolve $\kappa^2 \to \mu^2$, with an emission at κ^2.

- This interpretation will be helpful when thinking about time-like evolution and parton showering (i.e., in MCs)

Status – Parton Model + pQCD

- Basic structure of parton model remains valid, but distributions no longer scale precisely - there is a dimensionful quantity, Λ_{QCD}
- QCD coupling is small at short distance, large at large distance (as desired to explain the parton model) due to the short distance (UV) structure of the theory, i.e., physics at scales < $1/\mu$
- Can factor the complicated (hard to calculate) long distance, confining behavior from the short distance perturbative behavior at arbitrary factorization scale μ_F
- Determine the long distance behavior experimentally
- Perturbation theory predicts the short distance evolution, the IRS factors

Extra Detail Slides