QCD HW III

1. As discussed in the Lecture the parton distributions do not scale as in the naïve parton model but rather are expected to exhibit the scaling violation predicted by QCD. The structure of the expected renormalization of the parton distribution functions is summarized in terms of the DGLAP (also often called the "Altarelli-Parisi", *i.e.*, the AP in DGLAP) splitting functions. As noted in Lecture 3 the lowest order expressions for these functions are given by

$$P_{qq}^{(0)}(x) = C_F\left[\frac{1+x^2}{(1-x)_+} + \frac{3}{2}\delta(1-x)\right],$$

$$P_{qg}^{(0)}(x) = T_R\left[x^2 + (1-x)^2\right],$$

$$P_{gq}^{(0)}(x) = C_F\left[\frac{1+(1-x)^2}{x}\right],$$

$$P_{gg}^{(0)}(x) = 2\,C_A\left[\frac{x}{(1-x)_+} + \frac{(1-x)}{x} + x(1-x)\right] + \delta(1-x)\frac{11C_A - 4n_f T_R}{6},$$

where the "+" notation means

$$\int_0^1 dx\frac{f(x)}{(1-x)_+} \equiv \int_0^1 dx\frac{f(x)-f(1)}{(1-x)}.$$

a) Verify that the corresponding anomalous dimensions (the moments of these functions, $\gamma(j) \equiv \int_0^1 dx x^{j-1} P(x)$) have the forms

$$\gamma_{qq}^{(0)}(j) = C_F\left[-\frac{1}{2} + \frac{1}{j(j+1)} - 2\sum_{k=2}^{j}\frac{1}{k}\right],$$

$$\gamma_{qg}^{(0)}(j) = T_R\left[\frac{2+j+j^2}{j(j+1)(j+2)}\right],$$

$$\gamma_{gq}^{(0)}(j) = C_F \left[\frac{2+j+j^2}{j(j^2-1)}\right],$$

$$\gamma_{gg}^{(0)}(j) = 2C_A\left[-\frac{1}{12} + \frac{1}{j(j-1)} + \frac{1}{(j+1)(j+2)} - \sum_{k=2}^{j}\frac{1}{k}\right] - \left(\frac{2}{3}\right) n_f T_R.$$

Now consider the evolution of the singlet quark distribution given by the sum

$$\Sigma(x) = \sum_i q_i(x) + \bar{q}_i(x),$$

which mixes with the gluon distribution via the evolution equation. In terms of the moments with evolution variable $t = \ln\left(Q^2/\Lambda_{QCD}^2\right)$ we have

$$\frac{d}{dt}\begin{pmatrix}\Sigma(j)\\ g(j)\end{pmatrix} = \frac{\alpha_s(t)}{2\pi}\begin{pmatrix}\gamma_{qq}(j) & 2n_f\gamma_{qg}(j)\\ \gamma_{gq}(j) & \gamma_{gg}(j)\end{pmatrix}\begin{pmatrix}\Sigma(j)\\ g(j)\end{pmatrix}.$$

b) Verify that for j = 2 there are two eigenvalues to the above evolution equation and that the corresponding anomalous dimensions are $\lambda_+ = 0$ (momentum conservation) and $\lambda_- = -\left(16/9 + n_f/3\right)$ corresponding to the eigenvectors $\Sigma(2) + g(2)$ and $\Sigma(2) - 3n_f g(2)/16$, respectively.

c) Use the result of b) to find the momentum fraction in quarks and that in gluons at truly asymptotic values Q^2.

2. The evolution of the distribution functions tends to build up the gluon distribution at small *x*, which will be important at the LHC. Here we consider this point in more detail. In the limit of small *x* and very large Q^2 the DGLAP equation is dominated by the small argument behavior of the splitting function P_{gg}.

a) Verify that in this limit the gluon distribution $G(x,t) = xg(x,t)$ satisfies the equation (again $t = \ln\left(Q^2/\Lambda_{QCD}^2\right)$)

$$\frac{dG(x,t)}{dt} \simeq \frac{3\alpha_s(t)}{\pi}\int_x^1 \frac{dy}{y} G(y,t).$$

b) Now use the 1-loop form for α_s and change variables to $\tau = \ln t$ and $\varsigma = (24/b_0)\ln(1/x)$ to show that the approximate equation we want to solve is

$$\frac{d^2G(\varsigma,\tau)}{d\varsigma\, d\tau} \simeq \frac{1}{2} G(\varsigma,\tau).$$

c) Verify that at truly large values of both ζ and τ this equation is solved by

$$G(\varsigma,\tau) \sim e^{\sqrt{2\varsigma\tau}}$$

or

$$g(x,t) \sim \frac{1}{x}\exp\sqrt{\frac{48}{b_0}\ln\left(\frac{t}{t_0}\right)\ln\left(\frac{1}{x}\right)} \times \left(xg(x,t_0)\right).$$

d) To get a feeling for the size of this enhancement assume that the gluon distribution at Q_0 = 5 GeV is given by the following (fictitious) expression,

$$g(x) = \frac{420}{99}\frac{(1-x)^7}{x}.$$

Note that it already has the $1/x$ behavior at small x. Now evaluate the above enhancement factor for Q = 100 GeV at x = 0.01 with Λ_{QCD} = 0.1 GeV. How much larger is the evolved distribution at this x value, assuming that the above expression is relevant in the specified kinematic regime? (Take n_f = 5 for this estimate.)