QCD HW II

1. Using the Young diagram techniques mentioned in the Lecture find the SU(3) representation decomposition of $8 \otimes 3 = ?$ (e.g., an octet and a triplet, a gluon and a quark)

Do one of the following more challenging exercises.

2. We want to make use of the 4-D cross section for $e^+e^- \to q\bar{q}g$ presented in the Lecture

$$\frac{d\sigma}{dx_1 dx_2} = \sigma_0 \frac{\alpha_s}{2\pi} C_F \frac{x_1^2 + x_2^2}{(1-x_1)(1-x_2)},$$

where we have defined the dimensionless variables

$$x_i = \frac{2E_i}{\sqrt{q^2}} = \frac{2\, p_i \cdot q}{q^2} \Rightarrow x_i \geq 0 \Rightarrow \sum_{i=1}^{3} x_i = 2,$$

$$1 - \cos\theta_{ij} = \frac{p_i \cdot p_j}{E_i E_j} = \frac{(q - p_k)^2}{2E_i E_j} = \frac{2(1-x_k)}{x_i x_j}$$

with i = 1 = quark, i = 2 = anti-quark, i= 3 = gluon. The lowest order cross section (with just a quark and anti-quark in the final state is given by

$$\sigma_0 = \frac{4\pi\alpha^2}{3s} \cdot \sum_{\text{color}} \cdot \sum_f e_f^2 = \frac{4\pi\alpha^2}{3s} \cdot 3\sum_f e_f^2 .$$

If you have time, you should verify, by explicit calculation, these cross sections using the Feynman rules noted in the lectures.

a) As suggested in the lecture, it is straightforward to evaluate the Thrust distribution at this order in perturbation theory (from real emission). Recall that the quantity Thrust is defined by

$$T_3\left(p_q, p_{\bar{q}}, p_g\right) \equiv \frac{\max_{\hat{u}} \sum_{i=1,3} \left|\vec{p}_i \cdot \hat{u}\right|}{\sum_{i=1,3} \left|\vec{p}_i\right|},$$

with the NLO (next-to-leading order) Thrust distribution defined by

$$\frac{1}{\sigma}\frac{d\sigma}{dT} = \frac{1}{\sigma_0}\iint dx_1 dx_2 \frac{d\sigma}{dx_1 dx_2}\delta\left(T - T_3\left(x_i\right)\right).$$

The LO and virtual results contribute only at $T = 1$. Here we consider only the 3-body contribution. Use the definition of Thrust to verify that for any allowed configuration of the quark, anti-quark and gluon, the value of the thrust corresponds to the maximum x value,

$$T_3\left(p_q, p_{\bar{q}}, p_g\right) = \max\left[x_1, x_2, x_3\right],$$

and that

$$\frac{2}{3} \leq T_3 \leq 1.$$

HINT: Recall that $\sum_i \vec{p}_i = 0$.

b) Next focus on the configurations where the quark has the highest energy, $x_1 > x_2, x_3$. Verify that this region of phase space corresponds to

$$T = x_1 \geq x_2 \geq 2\left(1 - T\right),$$

and makes the following contribution to the Thrust distribution,

$$\left.\frac{1}{\sigma}\frac{d\sigma}{dT}\right|_{x_1 > x_2, x_3} = \frac{\alpha_s C_F}{2\pi}\frac{1}{1-T}\left\{\left(T^2 + 1\right)\ln\left(\frac{2T - 1}{1 - T}\right) + \left(\frac{3}{2}T^2 - 7T + 4\right)\right\}.$$

c) Next consider the contribution from the configurations where the gluon is the most energetic parton,

$$T = x_3 = 2 - x_1 - x_2 \geq x_1, x_2 .$$

Verify that this region of phase space corresponds to

$$x_1 = 2 - T - x_2, T \geq x_2 \geq 2(1-T),$$

and makes the following contribution to the Thrust distribution,

$$\left.\frac{1}{\sigma}\frac{d\sigma}{dT}\right|_{x_3 > x_1, x_2} = \frac{\alpha_s C_F}{2\pi}\frac{2}{T}\left\{\left(T^2 - 2T + 2\right)\ln\left(\frac{2T-1}{1-T}\right) + T\left(2-3T\right)\right\}.$$

d) Pull the 3 pieces (x_1, x_2 or x_3 as maximum) together to verify that the complete order α_s Thrust (away from 1) distribution is given by

$$\frac{1}{\sigma}\frac{d\sigma}{dT} = \frac{\alpha_s C_F}{2\pi}\left\{\frac{6T^2 - 6T + 4}{T\left(1-T\right)}\ln\left(\frac{2T-1}{1-T}\right) - \frac{3\left(2-T\right)\left(3T-2\right)}{1-T}\right\}.$$

Note that this expression verifies the result stated in class for the leading behavior in the limit $T \to 1$,

$$\frac{1}{\sigma}\frac{d\sigma}{dT}\xrightarrow[T\to 1]{}\frac{\alpha_s C_F}{2\pi}\left\{\frac{4}{\left(1-T\right)}\ln\left(\frac{1}{1-T}\right)\right\}.$$

3. Now we want to try one calculation in 4-2ε dimensions. Part of the challenge is to calculate the matrix element in the continued dimensions and to work out the changes in phase space. Here we will just accept the integral noted in the lecture and simply verify that the resulting contribution to the cross section is finite (for $\varepsilon < 0$). In fact, we will focus on just the terms that are singular as $\varepsilon \to 0$, although you are

encouraged to think about obtaining the finite bits. We are told that

$$\sigma^{q\bar{q}g}(\varepsilon) = \sigma_0 H(\varepsilon) \int dx_1 dx_2 \, \frac{C_F \alpha_s(\mu)}{2\pi} \times \left[\frac{(1-\varepsilon)\left(x_1^2 + x_2^2\right) + 2\varepsilon\left(x_1 + x_2 - 1\right)}{\left(1-x_1\right)^{1+\varepsilon}\left(1-x_2\right)^{1+\varepsilon}\left(x_1 + x_2 - 1\right)^{\varepsilon}} - 2\varepsilon \right],$$

where the function H(ε) expresses the ε dependence of the overall factor – the Born cross section in 4-2ε dimensions. By explicit calculation verify that

$$\sigma^{q\bar{q}g}(\varepsilon) = \sigma_0 \frac{C_F \alpha_s(\mu)}{2\pi} H(\varepsilon) \left[\frac{2}{\varepsilon^2} + \frac{3}{\varepsilon} + \mathcal{O}\left(\varepsilon^0\right) \right].$$

HINTS: Knowledge of the (old) Beta function,

$$\int_0^1 dx x^{\alpha-1}(1-x)^{\beta-1} = B(\alpha,\beta) = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)},$$

and the associated Gamma function is (very) useful.