QCD, Colliders & Jets - HW I Solutions

1. Our discussion of the strong interactions at high energies begins with the quark/parton model. In this approximation we treat hadrons as composite states of quarks, but we ignore the details of how they are confined to the hadrons. We need only know that we can describe hadrons in terms of their quark (and eventually gluon) content and that quarks produced in short distance, large momentum transfer interactions will eventually evolve into hadrons with probability 1 (and do it on a distance scale small compared to the size of detectors). We also assume that these short distance interactions of quarks (and leptons) are accurately described by the Lagrangians of $SU(3)_{QCD}$ x $SU(2)_L$ x $U(1)_Y$. One of the simplest applications is to the annihilation process of electrons and positrons at high energy. As an introduction we want to calculate the total cross section for $e^+e^- \to$ hadrons in the naive quark/parton model where the $e^+e^- \to q\bar{q}$ cross section (for a specific quark antiquark pair) is (assumed to be) identical to that for $e^+e^- \to \mu^+\mu^-$ except for the change in the electric charge. Evaluate your result at Q = 2, 7 and 15 GeV where Q is the total e^+e^- center of mass energy. Compare your results to data (see, *e.g.*, many of the recommended texts or the PDG Reviews) and comment. [HINT: Remember to sum over all appropriate quantum numbers.]

Solution: Starting with the cross section to muons and assuming that the only change is the sum over electric charges squared of the "active" quarks (*i.e.*, no color yet) we have (e_q is the charge of the quark in units of e)

$$\sigma\left(e^+e^- \to \text{hadrons}\right) \simeq \sigma\left(e^+e^- \to \mu^+\mu^-\right)\sum_q e_q^2$$
$$= \frac{4\pi\alpha^2}{3s}\sum_q e_q^2.$$

To get a sense of the size of these cross sections we note that

$$\frac{4\pi\alpha^2}{3\left(1\text{ GeV}\right)^2} \simeq 0.87\times10^{-4}\text{ mb.}$$

At 2 GeV the u, d and s quarks are active. At 7 GeV we include also the c quark and at 15 GeV we include the b quark. Thus we can make the following table.

Energy (GeV)	$\sum_q e_q^2$	σ
2	2/3	~ 14 nb
7	10/9	~ 2 nb
15	11/9	~ 0.5 nb

A typical way to present the data on electron-positron annihilation to hadrons is in terms of the ratio

$$R \equiv \frac{\sigma\left(e^+e^- \to \text{hadrons}\right)}{\sigma\left(e^+e^- \to \mu^+\mu^-\right)} \simeq \sum_q e_q^2,$$

which scales out the bulk of the variation with Q^2. Again we can make a table to compare the naïve parton model and the data. In this case we will include also the QCD improved result with a factor of 3 for the sum over color

$$\sigma\left(e^+e^- \to \text{hadrons, with color}\right) \simeq \sigma\left(e^+e^- \to \mu^+\mu^-\right) \sum_{\text{color}} \sum_q Q_q^2$$

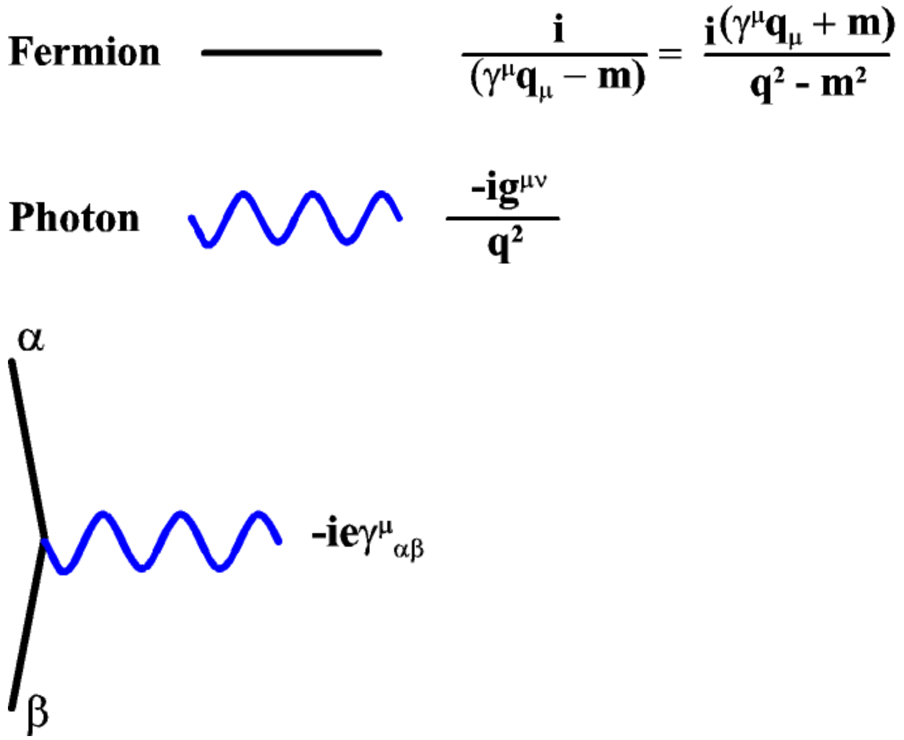
$$= \frac{4\pi\alpha^2}{s} \sum_q Q_q^2.$$

The new table looks like the following.

Energy (GeV)	R_{TH}	R_{data}	R_{color}
2	2/3	~ 1.6	2
7	10/9	~ 3.3	10/3
15	11/9	~ 3.7	11/3

Clearly even the “color” result is not perfect, which is no surprise at such low energies (we are making no effort to include the effects of the resonances in this energy region – the model should only work “on average”), but including color is clearly better than the situation without color.

2. As a first exercise let us recall how to find the cross section for (LO) elastic $e\mu$ scattering. The Feynman rules (in my notation) for the relevant propagators and vertices are.



a) With this starting point evaluated the appropriate spin averaged matrix element squared, $\left|\bar{\mathcal{M}}_{e^-\mu^-\to e^-\mu^-}\right|^2$, in terms of invariant quantities using the properties of the spinors and standard trace identities, *e.g.*,

$$\left(\not{p}-m\right)u\left(s,p\right)=0:\bar{u}\left(s,p\right)\left(\not{p}-m\right)=0 \text{ (fermions),}$$

$$\left(\not{p}+m\right)\mathrm{v}\left(s,p\right)=0:\bar{\mathrm{v}}\left(s,p\right)\left(\not{p}+m\right)=0 \text{ (antifermions),}$$

$$\sum_s u\left(s,p\right)\bar{u}\left(s,p\right)=\not{p}+m, \sum_s \mathrm{v}\left(s,p\right)\bar{\mathrm{v}}\left(s,p\right)=\not{p}-m,$$

yields

$$L_{\mu\nu} \equiv \frac{1}{2}\sum_{\mathrm{s_a},s_b} \bar{u}\left(s_b, p_b\right)\gamma_\mu u\left(s_a, p_a\right)\bar{u}\left(s_a, p_a\right)\gamma_\nu u\left(s_b, p_b\right)$$

$$= \frac{1}{2}\mathrm{Tr}\left[\left(\not{p}_b + m\right)\gamma_\mu\left(\not{p}_a + m\right)\gamma_\nu\right].$$

$$= 2\left[p_{a\mu}p_{b\nu} + p_{a\nu}p_{b\mu} - g_{\mu\nu}\left(p_a\cdot p_b - m^2\right)\right],$$

$$\mathrm{Tr}\left[\gamma^\alpha\gamma^\beta\gamma^\mu\gamma^\nu\right] = 4\left[g^{\alpha\beta}g^{\mu\nu} - g^{\alpha\mu}g^{\beta\nu} + g^{\alpha\nu}g^{\beta\mu}\right]$$

$$\mathrm{Tr}\left[\gamma^\alpha\gamma^\beta\right] = 4\left[g^{\alpha\beta}\right].$$

Solution: We start with the definition of the amplitude squared and spin summed with the notation a+b → 1+2

$$\left|\bar{\mathcal{M}}_{e^-\mu^-\to e^-\mu^-}\right|^2 = \left(\frac{1}{2}\right)^2\frac{\mathrm{e}^4}{t^2}\sum_{\mathrm{s_a},s_1}\bar{u}_{(e)}\left(s_1, p_1\right)\gamma_\mu u_{(e)}\left(s_a, p_a\right)\bar{u}_{(e)}\left(s_a, p_a\right)\gamma_\nu u_{(e)}\left(s_1, p_1\right)$$

$$\times\sum_{s_b,s_2}\bar{u}_{(\mu)}\left(s_2, p_2\right)\gamma^\mu u_{(\mu)}\left(s_b, p_b\right)\bar{u}_{(\mu)}\left(s_b, p_b\right)\gamma^\nu u_{(\mu)}\left(s_2, p_2\right)$$

$$\equiv\left(\frac{4\pi\alpha}{t}\right)^2 L^{(e)}_{\mu\nu}L^{(\mu)\mu\nu}\ .$$

$$\left|\bar{\mathcal{M}}_{e^-\mu^-\to e^-\mu^-}\right|^2 = 4\left(\frac{4\pi\alpha}{t}\right)^2\Big[2\left(p_a\cdot p_b\;\, p_1\cdot p_2 + p_a\cdot p_2\;\, p_b\cdot p_1\right)$$

$$-2\left(p_a\cdot p_1\right)\left(p_b\cdot p_2 - m_\mu^2\right) - 2\left(p_b\cdot p_2\right)\left(p_a\cdot p_1 - m_e^2\right)$$

$$+4\left(p_a\cdot p_1 - m_e^2\right)\left(p_b\cdot p_2 - m_\mu^2\right)\Big]$$

$$= \frac{32\pi^2\alpha^2}{t^2}\left[s^2+u^2-4\left(s+u\right)\left(m_e^2+m_\mu^2\right)+6\left(m_e^2+m_\mu^2\right)^2\right]$$

$$\simeq \frac{32\pi^2\alpha^2}{t^2}\left[s^2+u^2-4\left(s+u\right)\left(m_\mu^2\right)+6\left(m_\mu^2\right)^2\right]_{m_\mu\gg m_e}$$

$$\simeq \frac{32\pi^2\alpha^2}{t^2}\left[s^2+u^2\right]_{s\gg m_\mu^2}.$$

b) Find the corresponding invariant cross section, differential in the momentum transfer *t*, and the angular differential cross section evaluated in the “laboratory” frame (the rest frame of the muon) and compare to the expressions in the lecture.

$$\frac{d\sigma}{dt} = \frac{\left|\bar{\mathcal{M}}_{e^-\mu^-\to e^-\mu^-}\right|^2}{64\pi s p_{CM}^2} = \frac{\left|\bar{\mathcal{M}}_{e^-\mu^-\to e^-\mu^-}\right|^2}{16\pi\left(s-\left(m_e+m_\mu\right)^2\right)\left(s-\left(m_e-m_\mu\right)^2\right)}$$

$$\xrightarrow[s\gg m^2]{} \frac{\left|\bar{\mathcal{M}}_{e^-\mu^-\to e^-\mu^-}\right|^2}{16\pi s^2}$$

$$\frac{dt}{d\cos\theta_L} = \frac{2E^2}{\left(1+\frac{E}{m_\mu}\left(1-\cos\theta_L\right)\right)^2} = 2E'^2,$$

$$\frac{d\sigma}{d\Omega_L} = \frac{\left|\bar{\mathcal{M}}_{e^-\mu^-\to e^-\mu^-}\right|^2}{64\pi^2 m_\mu^2}\left(\frac{E'}{E}\right)^2 = \frac{\left|\bar{\mathcal{M}}_{e^-\mu^-\to e^-\mu^-}\right|^2}{64\pi^2 m_\mu^2}\left(\frac{1}{1+2\frac{E}{m_\mu}\sin^2 \theta_L/2}\right)^2.$$

Solution: Substituting the previous result for the amplitude we obtain, as desired, in the asymptotic limit

$$\left.\frac{d\sigma}{dt}\right|_{e^-\mu^-\to e^-\mu^-} \xrightarrow[s\gg m_\mu^2]{} \frac{2\pi\alpha^2}{s^2t^2}\left[s^2+u^2\right].$$

In the laboratory, or muon rest frame, we have

$$\begin{aligned}
s &= \left(p_a+p_b\right)^2 = m_e^2+m_\mu^2+2m_\mu E \simeq m_\mu^2+2m_\mu E \simeq 2m_\mu E,\\
t &= q^2 = \left(p_a-p_1\right)^2 = 2m_e^2-2p_a\cdot p_1 \simeq -2EE'\left(1-\cos\theta_L\right)\\
&\quad \simeq -4EE'\sin^2 \theta_L/2,\\
u &= \left(p_b-p_1\right)^2 = m_e^2+m_\mu^2-2p_b\cdot p_1 \simeq m_\mu^2-2m_\mu E'.
\end{aligned}$$

Requiring the structure muon to remain on-shell means that

$$\begin{aligned}
&p_2^2 = m_\mu^2 = \left(p_a+p_b-p_1\right)^2 = 2m_e^2+m_\mu^2+2p_a\cdot p_b-2p_a\cdot p_1-2p_1\cdot p_b\\
&\Rightarrow 0 \simeq 2Em_\mu-2E'm_\mu-2EE'\left(1-\cos\theta_L\right)\\
&\Rightarrow E' \simeq \frac{E}{1+\frac{E}{m_\mu}\left(1-\cos\theta_L\right)} = \frac{E}{1+2\frac{E}{m_\mu}\sin^2 \theta_L/2}.
\end{aligned}$$

So finally

$$\left.\frac{d\sigma}{d\Omega_L}\right|_{e\mu\to e\mu}=\frac{\alpha^2}{4E^2\sin^4\theta_L/2}\left(\frac{E'}{E}\right)\left[\cos^2\theta_L/2+\frac{2EE'}{m_\mu^2}\sin^4\theta_L/2\right]$$

$$=\frac{\alpha^2}{4E^2\sin^4\theta_L/2}\frac{\cos^2\theta_L/2}{1+\frac{2E}{m_\mu}\sin^2\theta_L/2}\left[1-\frac{q^2}{2m_\mu^2}\tan^2\theta_L/2\right].$$

3. Now we want to consider the elastic and inelastic scattering of an electron from a proton. Unlike the electron, we know that the proton is not an elementary particle. It has internal structure (the quarks and gluons) and the coupling to the photon will be more complicated than the simple electron vertex of the Feynman rules. However, we can use general Lorentz and symmetry considerations and the fact that the proton wave function must satisfy the Dirac equation to write the *elastic* coupling in terms of just 3 functions (form factors).

a) Use general Lorentz considerations and the Dirac equation to verify that the most general parity-conserving form of the electromagnetic current of the proton is

$$J^\mu\sim\bar{u}(p')\left[\Gamma_1(q^2)\gamma^\mu+\Gamma_2(q^2)i\sigma^{\mu\nu}q_\nu+\Gamma_3(q^2)q^\mu\right]u(p),$$

where $q^\mu=p'^\mu-p^\mu$ is the momentum of the photon ($q^2\neq 0$, in general) and, as usual, $\sigma^{\mu\nu}=\frac{i}{2}\left[\gamma^\mu,\gamma^\nu\right]$.

Solution: This is an application of the "only game in town" technique. The general form of the electromagnetic current due to the proton, *i.e.*, the lowest order coupling of the proton to photons must be

$$J^\mu\sim\bar{u}(p')O^\mu u(p),$$

i.e., there must be a spinor for the incoming proton and a barred spinor for the outgoing one (this guarantees the correct dependence on the proton spin) and O^μ must be an operator that transforms as a Lorentz vector. In order to have the correct parity, O^μ can have not factors of $\gamma5$ (or other pseudoscalars or pseudovectors). The (non-unique set of) available 4-vector operators with which to construct O^μ are γ^μ, $\sigma^{\mu\nu}q_\nu$ and q^μ. The symmetric combination $(p+p')^\mu$ is not independent since we have

$$\begin{aligned}\bar{u}(p')\left[i\sigma^{\mu\nu}q_\nu\right]u(p) &= -\tfrac{1}{2}\bar{u}(p')\left[\gamma^\mu,\left(\not{p}'-\not{p}\right)\right]u(p)\\ &= -\tfrac{1}{2}\bar{u}(p')\left[\gamma^\mu\left(\not{p}'-m\right)-\left(m-\not{p}\right)\gamma^\mu\right]u(p)\\ &= -\bar{u}(p')\left[(p'+p)^\mu-2m\gamma^\mu\right]u(p),\end{aligned}$$

where in the last step we used the Dirac equation for the proton spinors,

$$\left(\not{p}-m\right)u(p)=0,\bar{u}(p')\left(\not{p}'-m\right)=0.$$

Likewise the choice $\sigma^{\mu\nu}(p_\nu+p'_\nu)$ is also not independent, since

$$\begin{aligned}\bar{u}(p')\left[i\sigma^{\mu\nu}(p'+p)_\nu\right]u(p) &= -\tfrac{1}{2}\bar{u}(p')\left[\gamma^\mu,\left(\not{p}'+\not{p}\right)\right]u(p)\\ &= -\tfrac{1}{2}\bar{u}(p')\left[\gamma^\mu\left(\not{p}'+m\right)-\left(m+\not{p}\right)\gamma^\mu\right]u(p)\\ &= -\bar{u}(p')\left[\left(p'^\mu-p^\mu\right)\right]u(p)=-\bar{u}(p')\left[q^\mu\right]u(p).\end{aligned}$$

So the suggested set of 3 operators forms a complete (if not unique) choice. However, these considerations are not changed if we include the possibility of Lorentz invariant (scalar) coefficient functions, each a function of the external Lorentz scalars, p^2, p'^2, q^2. The first two are just m^2 and do not vary. Hence the only Lorentz invariant kinematic variable is q^2. Thus we are lead to the expression above as the most general form for the proton electromagnetic current.

b) What are the consequences of current conservation, $\partial^\mu J_\mu = 0$?

<u>Solution</u>: Current conservation says, in momentum space, that $q^\mu J_\mu = 0$,

$$\bar{u}(p')\left[\Gamma_1(q^2)\not{q} + \Gamma_2(q^2) i q_\mu \sigma^{\mu\nu} q_\nu + \Gamma_3(q^2) q^2\right] u(p) = 0.$$

But $\sigma^{\mu\nu}$ is antisymmetric so that $q_\mu \sigma^{\mu\nu} q_\nu = 0$ and

$$\begin{aligned}\bar{u}(p')\left[\not{q}\right]u(p) &= \bar{u}(p')\left[\not{p}' - \not{p}\right]u(p) \\ &= \bar{u}(p')\left[m - m\right]u(p) = 0.\end{aligned}$$

Thus current conservations boils down to the constraint

$$q^2 \Gamma_3(q^2) = 0,$$

which effectively means that Γ_3 vanishes, while Γ_1 and Γ_2 are unconstrained. Thus there are 2 possible scalar functions, or “form factors”, that describe the elastic interaction of photons with protons,

$$J^\mu \sim \bar{u}(p')\left[\Gamma_1(q^2)\gamma^\mu + \Gamma_2(q^2) i\sigma^{\mu\nu} q_\nu\right] u(p).$$

Now we want to consider the general form of *inelastic* electron-proton scattering, e p → e X, where a sum and integration over all possible physical states in X is implied. The connection to our previous study of the exclusive amplitude comes from the optical theorem, which relates the square of an amplitude for an inclusive process (e.g., a + b → X), appropriately summed and integrated over the inclusive final state (X), and the imaginary part of the corresponding forward elastic amplitude (e.g., a + b → a + b). We should think of $W_{\mu\nu}$ (defined below) as the imaginary part of the forward scattering amplitude of a (virtual) photon and a proton.

The general the spin averaged matrix element squared can be written in the form (assuming that only photon exchange contributes)

$$\left|\bar{\mathcal{M}}\right|^2 = \frac{e^4}{\left(q^2\right)^2} L^{\mu\nu} W_{\mu\nu}$$

where q^μ is the photon 4-momentum. This definition arose from the nonrelativistic limit for the proton (appropriate for earlier e p scattering experiments) and a factor of m_p (the proton mass) was pulled out of the $W_{\mu\nu}$ compared to the usual relativistic expectations, i.e., the dimension of $W_{\mu\nu}$ is 1/GeV. The tensor $L_{\mu\nu}$ describes the coupling to the electron and is given as an example in part a) above. With these definitions, the total (virtual) photon – proton cross section (in the lab frame) looks like

$$\sigma_\lambda\left(\gamma^* p\right) = \frac{4\pi^2\alpha}{q_{\text{lab}}}\varepsilon^\mu\left(\lambda\right)\varepsilon^{*\nu}\left(\lambda\right)W_{\mu\nu},$$

where the ε's are the photon polarizations (helicity λ). Note the flux factor lacks the expected factor of m_p (downstairs) due to the choice in the normalization of W. The corresponding definition of the inclusive e p scattering cross section (*i.e.*, put back in the electron – photon vertex in $L_{\mu\nu}$ and the photon propagators) is

$$d\sigma = \frac{\pi}{E}\left|\bar{\mathcal{M}}\right|^2\frac{d^3p'}{\left(2\pi\right)^3 2E'} = \frac{\pi}{E}\frac{e^4}{Q^4}L^{\mu\nu}W_{\mu\nu}\frac{d^3p'}{\left(2\pi\right)^3 2E'}.$$

c) Use general considerations of Lorentz, parity and current conservation issues (again) to verify that the hadronic part can be written in terms of just *two* invariant functions, W_1 and W_2, of the invariants q^2 and $\nu = q\cdot p/m$. Here p^μ is the proton 4-momentum and m its mass. The following form defines the two functions. You will also need to know something about the symmetry properties of $L^{\mu\nu}$.

$$W_{\mu\nu} = -W_1\left(g_{\mu\nu} - \frac{q_\mu q_\nu}{q^2}\right) + \frac{W_2}{m^2}\left(p_\mu - q\cdot p\frac{q_\mu}{q^2}\right)\left(p_\nu - q\cdot p\frac{q_\nu}{q^2}\right).$$

<u>Solution</u>: Since the proton is unpolarized, we must construct $W_{\mu\nu}$ only from the available 4-vectors, p_μ and q_μ, and the invariant tensors $g_{\mu\nu}$ and $\varepsilon_{\mu\nu\alpha\beta}$. Thus we can write the most general structure in terms of the possible 6 tensors and corresponding form factors,

$$W_{\mu\nu} = g_{\mu\nu}W_1 + \frac{p_\mu p_\nu}{m_p^2}W_2 + \frac{\left(p_\mu q_\nu + p_\nu q_\mu\right)}{m_p^2}W_3$$
$$+\frac{\left(p_\mu q_\nu - p_\nu q_\mu\right)}{m_p^2}W_4 + \frac{q_\mu q_\nu}{m_p^2}W_5 + \varepsilon_{\mu\nu\alpha\beta}\frac{p^\alpha q^\beta}{m_p^2}W_6.$$

The factor of $1/m^2$ was included to allow all of the form factors to have the same dimension. The fact that EM conserves parity means that $L_{\mu\nu}$ is symmetric and that we can ignore the antisymmetric terms in $W_{\mu\nu}$, W_4 and W_6. Conservation of the neutral current requires that $q^\mu W_{\mu\nu} = q^\nu W_{\mu\nu} = 0$ or

$$q_\nu W_1 m_p^2 + p\cdot q\, p_\nu W_2 + \left(q\cdot p q_\nu + p_\nu q^2\right)W_3 + q^2 q_\nu W_5 = 0$$

for arbitrary p_μ and q_μ. Hence the coefficients of p_ν and q_ν in this equation must separately vanish,

$$p\cdot q W_2 + q^2 W_3 = 0 \Rightarrow W_3 = -\frac{p\cdot q}{q^2}W_2,$$
$$W_1 m_p^2 + q\cdot p W_3 + q^2 W_5 = 0 \Rightarrow W_5 = \left(\frac{p\cdot q}{q^2}\right)^2 W_2 - \frac{m_p^2}{q^2}W_1.$$

Substituting we have

$$W_{\mu\nu} = W_1\left(g_{\mu\nu} - \frac{q_\mu q_\nu}{q^2}\right) + \frac{W_2}{m_p^2}\left(p_\mu p_\nu - \frac{q\cdot p}{q^2}\left[p_\mu q_\nu + p_\nu q_\mu\right] + q_\mu q_\nu\left[\frac{q\cdot p}{q^2}\right]^2\right)$$
$$= W_1\left(g_{\mu\nu} - \frac{q_\mu q_\nu}{q^2}\right) + \frac{W_2}{m_p^2}\left(p_\mu - q_\mu\frac{q\cdot p}{q^2}\right)\left(p_\nu - q_\nu\frac{q\cdot p}{q^2}\right).$$

d) Verify that the resulting cross section has the following general form, both in invariant notation and in the laboratory frame (*i.e.*, rest frame of the proton). We define θ_L to be the laboratory angle of the scattered electron, E' to be its energy, E to be the incident electron energy and $Q^2 = -q^2$ (ignoring lepton masses).

$$\frac{d\sigma}{dE'd\Omega_{Lab}} = \frac{4\alpha^2 E'^2}{Q^4}\left[2W_1\left(Q^2,\nu\right)\sin^2\left(\frac{\theta_L}{2}\right) + W_2\left(Q^2,\nu\right)\cos^2\left(\frac{\theta_L}{2}\right)\right],$$

$$\frac{d\sigma}{dQ^2 d\nu} = \frac{4\pi\alpha^2}{Q^4}\frac{E'}{E}\left[2W_1\left(Q^2,\nu\right)\sin^2\left(\frac{\theta_L}{2}\right) + W_2\left(Q^2,\nu\right)\cos^2\left(\frac{\theta_L}{2}\right)\right].$$

Solution: We can evaluate the lepton tensor to find

$$\begin{aligned} L^{\mu\nu} &= \frac{1}{2}\mathrm{Tr}\left[\bar{u}\left(l'\right)\gamma^{\mu}u\left(l\right)\bar{u}\left(l\right)\gamma^{\nu}u\left(l'\right)\right] \\ &\cong 2\left[l^{\mu}l'^{\nu} + l'^{\mu}l^{\nu} - g^{\mu\nu}l\cdot l'\right]. \end{aligned}$$

In the laboratory frame we have the following relations between the kinematic quantities

$$\begin{aligned} &l\cdot p = Em_p,\ l'\cdot p = E'm_p, \\ &q\cdot p = \left(E-E'\right)m_p \equiv \nu m_p, \\ &q^2 = \left(l-l'\right)^2 = -2l\cdot l' = -4EE'\sin^2 \theta/2. \end{aligned}$$

Noting that current conservation gives $q_\mu L^{\mu\nu} = q_\nu L^{\mu\nu} = 0$, we can evaluate the required product to find ($-q^2 = Q^2$)

$$\begin{aligned} L^{\mu\nu}W_{\mu\nu} &= 2W_1Q^2 + W_2\left[2\frac{l\cdot p l'\cdot p}{m_p^2} - Q^2\right] \\ &= 4EE'\left[2W_1\sin^2\theta/2 + W_2\left(1-\sin^2\theta/2\right)\right] \\ &= 4EE'\left[2W_1\sin^2\theta/2 + W_2\cos^2\theta/2\right] \\ \left|\bar{\mathcal{M}}\right|^2 &= \frac{64\pi^2}{q^4}EE'\left[2W_1\sin^2\theta/2 + W_2\cos^2\theta/2\right]. \end{aligned}$$

Finally the definition of the cross section gives

$$\begin{aligned}\frac{d^2\sigma}{dE'd\Omega'} &= \frac{1}{16\pi^2}\frac{E'}{E}\left|\bar{\mathcal{M}}\right|^2 \\ &= \frac{4\alpha^2 E'^2}{Q^4}\left[2W_1\left(Q^2,\nu\right)\sin^2 \theta/2 + W_2\left(Q^2,\nu\right)\cos^2 \theta/2\right], \\ \frac{d^2\sigma}{dQ^2 d\nu} &= \frac{\pi}{EE'}\frac{d^2\sigma}{dE'd\Omega'} = \frac{1}{16\pi}\frac{1}{E^2}\left|\bar{\mathcal{M}}\right|^2 \\ &= \frac{4\pi\alpha^2}{Q^4}\frac{E'}{E}\left[2W_1\left(Q^2,\nu\right)\sin^2 \theta/2 + W_2\left(Q^2,\nu\right)\cos^2 \theta/2\right].\end{aligned}$$

4. We want to make use of the 4-D cross section for $e^+e^- \to q\bar{q}g$ presented in the Lecture

$$\frac{d\sigma}{dx_1 dx_2} = \sigma_0 \frac{\alpha_s}{2\pi} C_F \frac{x_1^2 + x_2^2}{\left(1-x_1\right)\left(1-x_2\right)},$$

where we have defined the dimensionless variables

$$x_i = \frac{2E_i}{\sqrt{q^2}} = \frac{2p_i \cdot q}{q^2} \Rightarrow \; x_i \geq 0 \; \Rightarrow \sum_{i=1}^{3} x_i = 2,$$

$$1-\cos\theta_{ij} = \frac{p_i \cdot p_j}{E_i E_j} = \frac{\left(q-p_k\right)^2}{2E_i E_j} = \frac{2\left(1-x_k\right)}{x_i x_j}$$

with *i* = 1 = quark, *i* = 2 = anti-quark, *i*= 3 = gluon. The lowest order cross section (with just a quark and anti-quark in the final state is given by

$$\sigma_0 = \frac{4\pi\alpha^2}{3s} \cdot \sum_{\text{color}} \cdot \sum_f e_f^2 = \frac{4\pi\alpha^2}{3s} \cdot 3\sum_f e_f^2.$$

If you have time, you should verify, by explicit calculation, these cross sections using the Feynman rules noted in the lectures.

a) As suggested in the lecture (really Lecture 2), it is straightforward to evaluate the Thrust distribution at this order in perturbation theory (from real emission). Recall that the quantity Thrust is defined by

$$T_3\left(p_q, p_{\bar{q}}, p_g\right) \equiv \frac{\max_{\hat{u}} \sum\limits_{i=1,3} \left|\vec{p}_i \cdot \hat{u}\right|}{\sum\limits_{i=1,3} \left|\vec{p}_i\right|},$$

with the NLO (next-to-leading order) Thrust distribution defined by

$$\frac{1}{\sigma}\frac{d\sigma}{dT} = \frac{1}{\sigma_0}\iint dx_1 dx_2 \frac{d\sigma}{dx_1 dx_2}\delta\left(T - T_3\left(x_i\right)\right).$$

The LO and virtual results contribute only at $T = 1$. Here we consider only the 3-body contribution. Use the definition of Thrust to verify that for any allowed configuration of the quark, anti-quark and gluon, the value of the thrust corresponds to the maximum x value,

$$T_3\left(p_q, p_{\bar{q}}, p_g\right) = \max\left[x_1, x_2, x_3\right],$$

and that

$$\frac{2}{3} \le T_3 \le 1.$$

HINT: Recall that $\sum\limits_i \vec{p}_i = 0$.

<u>Solution</u>: The obvious candidate choices for the unit vector $\hat{u}$ are the 3 unit vectors, $\hat{p}_q, \hat{p}_{\bar{q}}, \hat{p}_g$. By conservation of 3-momentum we know that $\sum\limits_i \vec{p}_i = 0$ and thus the component of the total vector along any direction vanishes. For example, if we project along the quark direction, we have

$$\left(\sum_i \vec{p}_i\right)\cdot \hat{p}_q = \frac{\sqrt{q^2}}{2}\left(x_1 + x_2\cos\theta_{12} + x_3\cos\theta_{13}\right) = 0.$$

On the other hand the definition of Thrust has absolute values and the (absolute values of the) projections all add up rather than cancel. So we want to pick the direction along which the absolute values of the projections are the largest. By inspection this is clearly the direction of the most energetic parton. For example, if the quark is the most energetic parton ($x_1 > x_2, x_3$), then it must be that both $\cos\theta_{12}$ and $\cos\theta_{13}$ are negative (*i.e.*, the two smaller 3-momenta must point into the hemisphere opposite the largest 3-momentum). Thus for a leading quark we have

$$x_1 = x_2\left|\cos\theta_{12}\right| + x_3\left|\cos\theta_{13}\right|,$$

and this corresponds to the maximum projection along any axis. For this situation we find that the Thrust is

$$\begin{aligned} T_3\left(x_1 > x_2, x_3\right) &= \left.\frac{\sum\limits_{i=1,3}\left|\vec{p}_i\cdot\hat{p}_q\right|}{\sum\limits_{i=1,3}\left|\vec{p}_i\right|}\right|_{x_1>x_2,x_3} \\ &= \frac{1}{2}\left\{x_1 + x_2\left|\cos\theta_{12}\right| + x_3\left|\cos\theta_{13}\right|\right\} \\ &= \frac{1}{2}\left\{x_1 + x_1\right\} = x_1 = x_{\text{MAX}}. \end{aligned}$$

This analysis works analogously for whichever parton is most energetic and we find the desired result

$$T_3\left(p_q, p_{\bar{q}}, p_g\right) = \max\left[x_1, x_2, x_3\right].$$

Since the x_i are bounded above by 1, so is T, corresponding to the back-to-back, collinear configuration (1 parton in one direction and 2 in the opposite direction). The smallest maximum x_i value arises in the symmetric “Mercedes Benz” configuration with the 3 parton momentum vectors of equal length, lying in a plane and separated by angles of 2π/3,

$$\mathrm{Min}\left[x_{\mathrm{MAX}}\right]=\left\{x_1=x_2=x_3\right\}=\frac{2}{3}.$$

Thus, as expected, $2/3\le T_3\le 1$ (although for more complicated final states, say 4 particles, T can be as small as 0.5).

b) Next focus on the configurations where the quark has the highest energy, $x_1 > x_2, x_3$. Verify that this region of phase space corresponds to

$$T=x_1\ge x_2\ge 2\left(1-T\right),$$

and makes the following contribution to the Thrust distribution,

$$\left.\frac{1}{\sigma}\frac{d\sigma}{dT}\right|_{x_1>x_2,x_3}=\frac{\alpha_s C_F}{2\pi}\frac{1}{1-T}\left\{\left(T^2+1\right)\ln\left(\frac{2T-1}{1-T}\right)+\left(\frac{3}{2}T^2-7T+4\right)\right\}.$$

Solution: For this configuration we have $T = x_1$ and x_2 is the useful integration variable. By definition $x_1\ge x_2$ and the more interesting constraint is

$$\begin{aligned}&T=x_1\ge x_3=2-x_1-x_2=2-T-x_2\\&\Rightarrow x_2\ge 2-2T=2\left(1-T\right)\\&\Rightarrow T\ge x_2\ge 2\left(1-T\right).\end{aligned}$$

Hence the corresponding contribution to the Thrust distribution is

$$\begin{aligned}
\left.\frac{1}{\sigma}\frac{d\sigma}{dT}\right|_{x_1>x_2,x_3} &= \frac{1}{\sigma_0}\iint dx_1 dx_2\, \frac{d\sigma}{dx_1 dx_2}\,\delta\left(T-x_1\right)\Theta\left(x_1-x_2\right)\Theta\left(x_1-x_3\right) \\
&= \frac{\alpha_s C_F}{2\pi}\int_{2(1-T)}^{T} dx_2\, \frac{T^2+x_2^2}{\left(1-T\right)\left(1-x_2\right)} \\
&= \frac{\alpha_s C_F}{2\pi}\int_{2(1-T)}^{T} dx_2\, \frac{T^2+\left(1-x_2\right)^2-2\left(1-x_2\right)+1}{\left(1-T\right)\left(1-x_2\right)} \\
&= \frac{\alpha_s C_F}{2\pi}\left\{\frac{T^2+1}{1-T}\ln\left.\frac{1}{1-x_2}\right|_{2(1-T)}^{T} - \frac{2}{1-T}\left[T-2\left(1-T\right)\right]\right. \\
&\qquad\qquad \left. + \frac{1}{1-T}\left[T-2\left(1-T\right)\right] - \frac{1}{2\left(1-T\right)}\left[T^2-4\left(1-T\right)^2\right]\right. \\
&= \frac{\alpha_s C_F}{2\pi}\left\{\frac{T^2+1}{1-T}\ln\left(\frac{2T-1}{1-T}\right)+\frac{1}{1-T}\left[\frac{3}{2}T^2-7T+4\right]\right\}.
\end{aligned}$$

c) Next consider the contribution from the configurations where the gluon is the most energetic parton,

$$T = x_3 = 2 - x_1 - x_2 \geq x_1, x_2.$$

Verify that this region of phase space corresponds to

$$x_1 = 2 - T - x_2,\, T \geq x_2 \geq 2\left(1-T\right),$$

and makes the following contribution to the Thrust distribution,

$$\left.\frac{1}{\sigma}\frac{d\sigma}{dT}\right|_{x_3>x_1,x_2} = \frac{\alpha_s C_F}{2\pi}\frac{2}{T}\left\{\left(T^2-2T+2\right)\ln\left(\frac{2T-1}{1-T}\right)+T\left(2-3T\right)\right\}.$$

<u>Solution</u>: The organization of the limits of phase space occurs just as in the previous section, except that x_1 and x_3 are interchanged, but the limits on x_2 do not

change, $T \geq x_2 \geq 2(1-T)$. The integral defining the corresponding contribution to the Thrust distribution is then

$$\left.\frac{1}{\sigma}\frac{d\sigma}{dT}\right|_{x_3>x_1,x_2} = \frac{1}{\sigma_0}\iint dx_1 dx_2\,\frac{d\sigma}{dx_1 dx_2}\,\delta(T-x_3)\,\Theta(x_3-x_1)\,\Theta(x_3-x_2)$$

$$= \frac{\alpha_s C_F}{2\pi}\int\limits_{2(1-T)}^{T} dx_2\,\frac{(2-T-x_2)^2+x_2^2}{(T+x_2-1)(1-x_2)}.$$

We can simplify this integrand by separating the two denominators and then integrate,

$$\left.\frac{1}{\sigma}\frac{d\sigma}{dT}\right|_{x_3>x_1,x_2} = \frac{\alpha_s C_F}{2\pi}\int\limits_{2(1-T)}^{T} dx_2\,\frac{(2-T-x_2)^2+x_2^2}{(T+x_2-1)(1-x_2)}$$

$$= \frac{\alpha_s C_F}{2\pi}\int\limits_{2(1-T)}^{T} dx_2\left\{\frac{T^2-2T+2}{T(T+x_2-1)}+\frac{T^2-2T+2}{T(1-x_2)}-2\right\}$$

$$= \frac{\alpha_s C_F}{2\pi}\left\{2\,\frac{T^2-2T+2}{T}\ln\left(\frac{2T-1}{1-T}\right)-6T+4\right\}.$$

d) Pull the 3 pieces (x_1, x_2 or x_3 as maximum) together to verify that the complete order α_s Thrust (away from 1) distribution is given by

$$\frac{1}{\sigma}\frac{d\sigma}{dT} = \frac{\alpha_s C_F}{2\pi}\left\{\frac{6T^2-6T+4}{T(1-T)}\ln\left(\frac{2T-1}{1-T}\right)-\frac{3(2-T)(3T-2)}{1-T}\right\}.$$

Note that this expression verifies the result stated in class for the leading behavior in the limit $T \to 1$,

$$\frac{1}{\sigma}\frac{d\sigma}{dT}\xrightarrow[T\to 1]{}\frac{\alpha_s C_F}{2\pi}\left\{\frac{4}{\left(1-T\right)}\ln\left(\frac{1}{1-T}\right)\right\}.$$

Solution: The "full" answer consists of adding together the contributions when the quark has the largest energy (part b), when the anti-quark has the largest energy (again part b due to the 1↔2 symmetry) and when the gluon has the largest energy (part c). This yields the desired result

$$\begin{aligned}\left.\frac{1}{\sigma}\frac{d\sigma}{dT}\right|_{\text{NLO}} &= \frac{\alpha_s C_F}{2\pi}\left\{2\frac{T^2+1}{1-T}\ln\left(\frac{2T-1}{1-T}\right)+\frac{2}{1-T}\left[\frac{3}{2}T^2-7T+4\right]\right.\\ &\qquad\left.+\frac{2}{T}\left(T^2-2T+2\right)\ln\left(\frac{2T-1}{1-T}\right)-6T+4\right\}\\ &= \frac{\alpha_s C_F}{2\pi}\left\{\frac{2\left(3T^2-3T+2\right)}{T\left(1-T\right)}\ln\left(\frac{2T-1}{1-T}\right)-\frac{3\left(3T-2\right)\left(2-T\right)}{1-T}\right\}.\end{aligned}$$

5. Now we want to try one calculation in 4-2ε dimensions. Part of the challenge is to calculate the matrix element in the continued dimensions and to work out the changes in phase space. Here we will just accept the integral noted in the lecture and simply verify that the resulting contribution to the cross section is finite (for ε < 0). In fact, we will focus on just the terms that are singular as ε → 0, although you are encouraged to think about obtaining the finite bits. We are told that

$$\sigma^{q\bar{q}g}\left(\varepsilon\right)=\sigma_0 H\left(\varepsilon\right)\frac{C_F\alpha_s\left(\mu\right)}{2\pi}\int_0^1 dx_1 dx_2\left[\frac{\left(1-\varepsilon\right)\left(x_1^2+x_2^2\right)+2\varepsilon\left(x_1+x_2-1\right)}{\left(1-x_1\right)^{1+\varepsilon}\left(1-x_2\right)^{1+\varepsilon}\left(x_1+x_2-1\right)^{\varepsilon}}-2\varepsilon\right],$$

where the function H(ε) expresses the ε dependence of the overall factor – the Born cross section in 4-2ε dimensions. By explicit calculation verify that

$$\sigma^{q\bar{q}g}\left(\varepsilon\right)=\sigma_0\frac{C_F\alpha_s\left(\mu\right)}{2\pi}H\left(\varepsilon\right)\left[\frac{2}{\varepsilon^2}+\frac{3}{\varepsilon}+\mathcal{O}\left(\varepsilon^0\right)\right].$$

HINTS: Knowledge of the (old) Beta function,

$$\int_0^1 dx x^{\alpha-1}(1-x)^{\beta-1} = B(\alpha,\beta) = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)},$$

and the associated Gamma function is (very) useful.

<u>Solution</u>: This is a simple exercise if you are facile with Gamma/Beta functions and you can convince yourself that the factor $(x_1+x_2-1)^\varepsilon$ does not affect the singular bits. Note in particular that this is just $(1-x_3)^\varepsilon$ and the region $x_3 \to 1$ is not singular. Ignoring this factor we have

$$\begin{aligned}
&\int_0^1 dx_1 dx_2 \left[\frac{(1-\varepsilon)\left(x_1^2+x_2^2\right)+2\varepsilon(x_1+x_2-1)}{(1-x_1)^{1+\varepsilon}(1-x_2)^{1+\varepsilon}(x_1+x_2-1)^\varepsilon} - 2\varepsilon\right] \\
&\quad \simeq \int_0^1 dx_1 dx_2 \left[\frac{(1-\varepsilon)\left(x_1^2+x_2^2\right)+2\varepsilon(x_1+x_2-1)}{(1-x_1)^{1+\varepsilon}(1-x_2)^{1+\varepsilon}}\right] \\
&\quad = (1-\varepsilon)\int_0^1 dx_1 dx_2 \frac{x_1^2+x_2^2}{(1-x_1)^{1+\varepsilon}(1-x_2)^{1+\varepsilon}} - 2\varepsilon\int_0^1 dx_1 dx_2 \frac{(1-x_1)+(1-x_2)-1}{(1-x_1)^{1+\varepsilon}(1-x_2)^{1+\varepsilon}} \\
&\quad = (1-\varepsilon)2B(3,-\varepsilon)B(1,-\varepsilon) - 4\varepsilon B(1,1-\varepsilon)B(1,-\varepsilon) + 2\varepsilon B^2(1,-\varepsilon) \\
&\quad = (1-\varepsilon)2\frac{\Gamma(3)\Gamma(-\varepsilon)}{\Gamma(3-\varepsilon)}\frac{\Gamma(1)\Gamma(-\varepsilon)}{\Gamma(1-\varepsilon)} - 4\varepsilon\frac{\Gamma(1)\Gamma(1-\varepsilon)}{\Gamma(2-\varepsilon)}\frac{\Gamma(1)\Gamma(-\varepsilon)}{\Gamma(1-\varepsilon)} + 2\varepsilon\left(\frac{\Gamma(1)\Gamma(-\varepsilon)}{\Gamma(1-\varepsilon)}\right)^2 \\
&\quad = \frac{(1-\varepsilon)}{(-\varepsilon)^2}2\frac{\Gamma(3)\Gamma(1-\varepsilon)}{(2-\varepsilon)(1-\varepsilon)\Gamma(1-\varepsilon)}\frac{\Gamma(1-\varepsilon)}{\Gamma(1-\varepsilon)} - 4\frac{\varepsilon}{-\varepsilon}\frac{\Gamma(1)\Gamma(1-\varepsilon)}{(1-\varepsilon)\Gamma(1-\varepsilon)}\frac{\Gamma(1)\Gamma(-\varepsilon)}{\Gamma(1-\varepsilon)} \\
&\quad + 2\frac{\varepsilon}{(-\varepsilon)^2}\left(\frac{\Gamma(1)\Gamma(1-\varepsilon)}{\Gamma(1-\varepsilon)}\right)^2
\end{aligned}$$

$$=\frac{(1-\varepsilon)}{(-\varepsilon)^2}\frac{4}{(2-\varepsilon)(1-\varepsilon)}+\frac{4}{1-\varepsilon}+\frac{2}{\varepsilon}\simeq\frac{2}{\varepsilon^2}(1-\varepsilon)\left(1+\frac{\varepsilon}{2}\right)(1+\varepsilon)+\frac{2}{\varepsilon}+\mathcal{O}\left(\varepsilon^0\right)$$

$$=\frac{2}{\varepsilon^2}+\frac{3}{\varepsilon}+\mathcal{O}\left(\varepsilon^0\right).$$