

QCD, Colliders & Jets - HW II Solutions

1. As discussed in the Lecture the parton distributions do not scale as in the naïve parton model but rather are expected to exhibit the scaling violation predicted by QCD. The structure of the expected renormalization of the parton distribution functions is summarized in terms of the DGLAP (also often called the “Altarelli-Parisi”, *i.e.*, the AP in DGLAP) splitting functions. As noted in Lecture 2 the lowest order expressions for these functions are given by

$$\begin{aligned}
 P_{qq}^{(0)}(x) &= C_F \left[\frac{1+x^2}{(1-x)_+} + \frac{3}{2} \delta(1-x) \right], \\
 P_{qg}^{(0)}(x) &= T_R \left[x^2 + (1-x)^2 \right], \\
 P_{gq}^{(0)}(x) &= C_F \left[\frac{1+(1-x)^2}{x} \right], \\
 P_{gg}^{(0)}(x) &= 2C_A \left[\frac{x}{(1-x)_+} + \frac{(1-x)}{x} + x(1-x) \right] + \delta(1-x) \frac{11C_A - 4n_f T_R}{6},
 \end{aligned}$$

where the “+” notation means

$$\int_0^1 dx \frac{f(x)}{(1-x)_+} \equiv \int_0^1 dx \frac{f(x) - f(1)}{(1-x)}.$$

a) Verify that the corresponding anomalous dimensions (the moments of these functions, $\gamma(j) \equiv \int_0^1 dx x^{j-1} P(x)$) have the forms

$$\begin{aligned}
 \gamma_{qq}^{(0)}(j) &= C_F \left[-\frac{1}{2} + \frac{1}{j(j+1)} - 2 \sum_{k=2}^j \frac{1}{k} \right], \\
 \gamma_{qg}^{(0)}(j) &= T_R \left[\frac{2+j+j^2}{j(j+1)(j+2)} \right],
 \end{aligned}$$

$$\gamma_{gq}^{(0)}(j) = C_F \left[\frac{2+j+j^2}{j(j^2-1)} \right],$$

$$\gamma_{gg}^{(0)}(j) = 2C_A \left[-\frac{1}{12} + \frac{1}{j(j-1)} + \frac{1}{(j+1)(j+2)} - \sum_{k=2}^j \frac{1}{k} \right] - \left(\frac{2}{3} \right) n_f T_R.$$

Solution: This is a straightforward exercise to performing the appropriate integrals. The anomalous dimension for the quark to quark process has the form

$$\begin{aligned} \gamma_{qq}^{(0)}(j) &= C_F \int_0^1 dz z^{j-1} \left[\frac{1+z^2}{(1-z)_+} + \frac{3}{2} \delta(1-z) \right] \\ &= C_F \int_0^1 dz z^{j-1} \left(\frac{1+z^2}{1-z} \right)_+ \\ &= C_F \int_0^1 dz \left(\frac{1+z^2}{1-z} \right) (z^{j-1} - 1) \\ &= C_F \int_0^1 dz \left\{ \left(\sum_{k=j-1}^{\infty} z^k - \sum_{k=0}^{\infty} z^k \right) + \left(\sum_{k=j+1}^{\infty} z^k - \sum_{k=2}^{\infty} z^k \right) \right\} \\ &= C_F \int_0^1 dz \left\{ -\sum_{k=0}^{j-2} z^k - \sum_{k=2}^j z^k \right\} \\ &= C_F \left\{ -\sum_{k=1}^{j-1} \frac{1}{k} - \sum_{k=3}^{j+1} \frac{1}{k} \right\} \\ &= C_F \left\{ -2 \sum_{k=2}^j \frac{1}{k} - 1 + \frac{1}{2} + \frac{1}{j} - \frac{1}{j+1} \right\} \\ &= C_F \left[-\frac{1}{2} + \frac{1}{j(j+1)} - 2 \sum_{k=2}^j \frac{1}{k} \right]. \end{aligned}$$

For the gluon to quark case we have

$$\begin{aligned}
\gamma_{qg}^{(0)}(j) &= T_R \int_0^1 dz z^{j-1} \left[z^2 + (1-z)^2 \right] \\
&= T_R \int_0^1 dz \left[z^{j+1} + z^{j-1} - 2z^j + z^{j+1} \right] \\
&= T_R \left[\frac{2}{j+2} + \frac{1}{j} - \frac{2}{j+1} \right] \\
&= T_R \left[\frac{2j(j+1) + (j+1)(j+2) - 2j(j+2)}{j(j+1)(j+2)} \right] \\
&= T_R \left[\frac{2+j+j^2}{j(j+1)(j+2)} \right].
\end{aligned}$$

Similarly for quark to gluon we find

$$\begin{aligned}
\gamma_{gq}^{(0)}(j) &= C_F \int_0^1 dz z^{j-1} \left[\frac{1+(1-z)^2}{z} \right] \\
&= C_F \int_0^1 dz \left[2z^{j-2} - 2z^{j-1} + z^j \right] \\
&= C_F \left[\frac{2}{j-1} - \frac{2}{j} + \frac{1}{j+1} \right] \\
&= C_F \left[\frac{2j(j+1) - 2(j-1)(j+1) + j(j-1)}{j(j-1)(j+1)} \right] \\
&= C_F \left[\frac{2+j+j^2}{j(j^2-1)} \right].
\end{aligned}$$

Finally for gluon to gluon we have

$$\begin{aligned}
\gamma_{gg}^{(0)}(j) &= \int_0^1 dz z^{j-1} \left\{ 2C_A \left[\frac{z}{(1-z)_+} + \frac{(1-z)}{z} + z(1-z) \right] + \delta(1-z) \frac{11C_A - 4n_f T_R}{6} \right\} \\
&= 2C_A \int_0^1 dz \left[\frac{z^j - 1}{1-z} + (1-z)z^{j-2} + z^j(1-z) \right] + \frac{11C_A - 4n_f T_R}{6} \\
&= 2C_A \int_0^1 dz \left[\sum_{k=0}^{\infty} z^{k+j} - \sum_{k=0}^{\infty} z^k + z^{j-2} - z^{j-1} + z^j - z^{j+1} \right] + \frac{11C_A - 4n_f T_R}{6} \\
&= 2C_A \left[-\sum_{k=1}^j \frac{1}{k} + \frac{1}{j-1} - \frac{1}{j} + \frac{1}{j+1} - \frac{1}{j+2} \right] + \frac{11C_A - 4n_f T_R}{6} \\
&= 2C_A \left[\frac{1}{j(j-1)} + \frac{1}{(j+1)(j+2)} - \sum_{k=1}^j \frac{1}{k} + \frac{11}{12} \right] - \frac{2n_f T_R}{3} \\
&= 2C_A \left[\frac{1}{j(j-1)} + \frac{1}{(j+1)(j+2)} - \sum_{k=2}^j \frac{1}{k} - \frac{1}{12} \right] - \frac{2n_f T_R}{3}.
\end{aligned}$$

Now consider the evolution of the singlet quark distribution given by the sum

$$\Sigma(x) = \sum_i q_i(x) + \bar{q}_i(x),$$

which mixes with the gluon distribution via the evolution equation. In terms of the moments with evolution variable $t = \ln(Q^2/\Lambda_{QCD}^2)$ we have

$$\frac{d}{dt} \begin{pmatrix} \Sigma(j) \\ g(j) \end{pmatrix} = \frac{\alpha_s(t)}{2\pi} \begin{pmatrix} \gamma_{qq}(j) & 2n_f \gamma_{qg}(j) \\ \gamma_{gq}(j) & \gamma_{gg}(j) \end{pmatrix} \begin{pmatrix} \Sigma(j) \\ g(j) \end{pmatrix}.$$

- b) Verify that for $j = 2$ there are two eigenvalues to the above evolution equation and that the corresponding anomalous dimensions are $\lambda_+ = 0$ (momentum

conservation) and $\lambda_- = -\left(16/9 + n_f/3\right)$ corresponding to the eigenvectors $\Sigma(2) + g(2)$ and $\Sigma(2) - 3n_f g(2)/16$, respectively.

Solution: First we need $\gamma(2)$ for the various channels. From the results of part a) we have ($C_F = 4/3$, $C_A = 3$, $T_R = 1/2$)

$$\gamma_{qq}(2) = \frac{4}{3} \left[-\frac{1}{2} + \frac{1}{6} - 2 \frac{1}{2} \right] = \frac{4}{3} \left[-\frac{3}{2} + \frac{1}{6} \right] = -\frac{16}{9}$$

$$\gamma_{qg}(2) = \frac{1}{2} \left[\frac{2+2+4}{24} \right] = \frac{1}{6}$$

$$\gamma_{gq}(2) = \frac{4}{3} \left[\frac{2+2+4}{6} \right] = \frac{16}{9}$$

$$\gamma_{gg}(2) = 6 \left[\frac{1}{2} + \frac{1}{12} - \frac{1}{2} - \frac{1}{12} \right] - \frac{n_f}{3} = -\frac{n_f}{3}.$$

Thus the “anomalous dimension matrix” is

$$A = \begin{pmatrix} -\frac{16}{9} & \frac{n_f}{3} \\ \frac{16}{9} & -\frac{n_f}{3} \end{pmatrix},$$

where $\det A = 0$ and $\text{Tr} A = -\left(\frac{16}{9} + \frac{n_f}{3}\right)$. Thus the eigenvalues must satisfy

$$\det A = \lambda_+ \lambda_- = 0, \text{Tr} A = \lambda_+ + \lambda_- = -\left(\frac{16}{9} + \frac{n_f}{3}\right).$$

We easily deduce that the eigenvalues, with the appropriate identification by magnitude, are

$$\lambda_+ = 0$$

$$\lambda_- = -\left(\frac{16}{9} + \frac{n_f}{3}\right).$$

If we write the eigenvectors in the form $\Sigma(2) + C_{\pm}g(2)$ (with C_{\pm} the constants that we want to solve for), we have the evolution equation

$$\frac{d}{dt}(\Sigma(2) + C_{\pm}g(2)) = \frac{\alpha_s}{2\pi} \lambda_{\pm} (\Sigma(2) + C_{\pm}g(2)).$$

Thus the matrix equations we must solve is

$$A^T \begin{pmatrix} 1 \\ C_{\pm} \end{pmatrix} = \begin{pmatrix} -\frac{16}{9} & \frac{16}{9} \\ \frac{n_f}{3} & -\frac{n_f}{3} \end{pmatrix} \begin{pmatrix} 1 \\ C_{\pm} \end{pmatrix} = \lambda_{\pm} \begin{pmatrix} 1 \\ C_{\pm} \end{pmatrix} = \begin{matrix} 0 \begin{pmatrix} 1 \\ C_+ \end{pmatrix} \\ -\left(\frac{16}{9} + \frac{n_f}{3}\right) \begin{pmatrix} 1 \\ C_- \end{pmatrix} \end{matrix}.$$

These are easy to solve and we find the desired results

$$C_+ = 1; C_- = \frac{-n_f/3}{16/9} = -\frac{3n_f}{16}.$$

Thus the first eigenfunction is $\Sigma(2) + g(2)$, *i.e.*, total momentum, and the eigenvalue, $\lambda_+ = 0$, just corresponds to total momentum conservation, *i.e.*, the total momentum does not change with the scaling parameter t (and is always equal to 1).

- c) Use the result of b) to find the momentum fraction in quarks and that in gluons at truly asymptotic values Q^2 .

Solution: We now use the results of part b) to consider the relative fraction of momentum in quarks and gluons at asymptotic scales. In particular, we want to

focus on the eigenfunction above with the negative eigenvalue. The general form of the solution to the DGLAP equation tells us that the eigenfunction behaves as

$$\begin{aligned} [\Sigma(2) + C_- g(2)](t) &= [\Sigma(2) + C_- g(2)](t_0) \left(\frac{\alpha_s(t_0)}{\alpha_s(t)} \right)^{\lambda_- \frac{2}{b_0}} \\ &\xrightarrow[t \rightarrow \infty]{} 0, \lambda_- < 0. \end{aligned}$$

The asymptotic momentum fractions can be obtained from this result, in the form

$$\Sigma(2, t \rightarrow \infty) = -C_- g(2, t \rightarrow \infty) = \frac{3n_f}{16} g(2, t \rightarrow \infty),$$

and from momentum conservation,

$$\Sigma(2, t \rightarrow \infty) = 1 - g(2, t \rightarrow \infty) = 1 - \frac{16}{3n_f} \Sigma(2, t \rightarrow \infty).$$

Assuming that the (truly) asymptotic value of n_f is 6, we have

$$\Sigma(2, t \rightarrow \infty) = \frac{1}{1 + \frac{16}{3n_f}} \Bigg|_{n_f=6} = \frac{9}{17} = 0.529,$$

$$g(2, t \rightarrow \infty) = \frac{8}{17} = 0.471,$$

$$\frac{\Sigma(2, t \rightarrow \infty)}{g(2, t \rightarrow \infty)} = \frac{9}{8} = 1.125.$$

At truly asymptotic scales we expect slightly more momentum in quarks than gluons in reasonable agreement with current data.

2. The evolution of the distribution functions tends to build up the gluon distribution at small x , which will be important at the LHC. Here we consider this point in more detail. In the limit of small x and very large Q^2 the DGLAP equation is dominated by the small argument behavior of the splitting function P_{gg} .

a) Verify that in this limit the gluon distribution $G(x,t) = xg(x,t)$ satisfies the equation (again $t = \ln(Q^2/\Lambda_{QCD}^2)$)

$$\frac{dG(x,t)}{dt} \simeq \frac{3\alpha_s(t)}{\pi} \int_x^1 \frac{dy}{y} G(y,t).$$

Solution: We start with the “standard” DGLAP result

$$\frac{d}{dt} G(x,t) = \frac{\alpha_s(t)}{2\pi} x \int dy \int dz \delta(x-yz) [P_{gq}(z)\Sigma(y,t) + P_{qq}(z)g(y,t)].$$

For $x \rightarrow 0$ the dominant contribution comes from soft gluon emission (recall the $1/z$ bits in the splitting functions) where both y and $z \rightarrow 0$. Hence the dominant soft gluon term evolves from the “previous” soft gluon terms and $\lim_{y \rightarrow 0} \frac{\Sigma(y)}{g(y)} \rightarrow 0$.

Also we have $P_{gg}(z) \xrightarrow{z \rightarrow 0} \frac{2C_A}{z} = \frac{6}{z}$. So we can write

$$\begin{aligned} \frac{d}{dt} G(x,t) &\simeq \frac{\alpha_s(t)}{2\pi} 6x \int \frac{dy}{y} \int \frac{dz}{z} \delta(x-yz) [yg(y,t)] \\ &\simeq 3 \frac{\alpha_s(t)}{\pi} x \int \frac{dy}{y} \int \frac{dz}{z} \delta(x-yz) [G(y,t)]. \end{aligned}$$

Now we can make use of the following property of the delta function

$$x \int \frac{dz}{z} \delta(x-zy) = x \frac{\theta(y-x)}{y \cdot \frac{x}{y}} = \theta(y-x).$$

Using this result in the previous equation yields the desired result in the small x limit

$$\left. \frac{d}{dt} G(x, t) \right|_{x \rightarrow 0} \simeq 3 \frac{\alpha_s(t)}{\pi} \int_x^1 \frac{dy}{y} G(y, t).$$

b) Now use the 1-loop form for α_s and change variables to $\tau = \ln t$ and $\zeta = (24/b_0) \ln(1/x)$ to show that the approximate equation we want to solve is

$$\frac{d^2 G(\zeta, \tau)}{d\zeta d\tau} \simeq \frac{1}{2} G(\zeta, \tau).$$

Solution: With the 1-loop form of the coupling we have

$$\frac{\alpha_s}{\pi} \simeq \frac{4}{b_0 t} = \frac{4}{b_0 e^\tau}.$$

Thus we can rewrite the previous result as

$$t \frac{d}{dt} G(x, t) = \frac{d}{d\tau} G(x, \tau) \simeq \frac{12}{b_0} \int_x^1 \frac{dy}{y} G(y, \tau),$$

and then taking a derivative with respect to x we find

$$\frac{\partial^2}{\partial \tau \partial x} G(x, \tau) \simeq -\frac{12}{b_0 x} G(x, \tau).$$

We can simplify this further with the variable defined above, $\zeta = (24/b_0) \ln(1/x)$, with $d\zeta = -(24/b_0)(dx/x)$. So finally we have the desired result

$$\frac{\partial^2}{\partial \tau \partial \zeta} G(\zeta, \tau) \approx \frac{1}{2} G(\zeta, \tau).$$

c) Verify that at truly large values of both ζ and τ this equation is solved by

$$G(\zeta, \tau) \sim e^{\sqrt{2\zeta\tau}}$$

or

$$g(x, t) \sim \frac{1}{x} \exp \sqrt{\frac{48}{b_0} \ln \left(\frac{t}{t_0} \right) \ln \left(\frac{1}{x} \right)} \times (xg(x, t_0)).$$

Solution: In the limit of large ζ and τ , i.e., $\ln t$ and $\ln 1/x$ large, we have

$$\begin{aligned} \frac{\partial}{\partial \zeta} e^{\sqrt{2\zeta\tau}} &= \frac{1}{2} \sqrt{\frac{2\tau}{\zeta}} e^{\sqrt{2\zeta\tau}}, \\ \frac{\partial}{\partial \tau} \frac{\partial}{\partial \zeta} e^{\sqrt{2\zeta\tau}} &= \frac{1}{2} \sqrt{\frac{2\tau}{\zeta}} \cdot \frac{1}{2} \sqrt{\frac{2\zeta}{\tau}} \cdot e^{\sqrt{2\zeta\tau}} + \frac{1}{2} \cdot \frac{1}{2} \sqrt{\frac{2}{\zeta\tau}} \cdot e^{\sqrt{2\zeta\tau}}, \\ \frac{\partial^2}{\partial \tau \partial \zeta} e^{\sqrt{2\zeta\tau}} \Big|_{\zeta, \tau \gg 1} &\approx \frac{1}{2} e^{\sqrt{2\zeta\tau}}. \end{aligned}$$

Thus, as desired, the asymptotic solution of the differential equation is proportional to

$$G(\zeta, \tau) \sim e^{\sqrt{2\zeta\tau}}.$$

So, substituting in terms of the original functions and variables and including an appropriate overall normalization, we have the asymptotic result

$$g(x, t) \approx \frac{1}{x} \exp \sqrt{\frac{48}{b_0} \ln \left(\frac{t}{t_0} \right) \ln \left(\frac{1}{x} \right)} \times (xg(x, t_0)).$$

d) To get a feeling for the size of this enhancement assume that the gluon distribution at $Q_0 = 5$ GeV is given by the following (fictitious) expression,

$$g(x) = \frac{420}{99} \frac{(1-x)^7}{x}.$$

Note that it already has the $1/x$ behavior at small x . Now evaluate the above enhancement factor for $Q = 100$ GeV at $x = 0.01$ with $\Lambda_{QCD} = 0.1$ GeV. How much larger is the evolved distribution at this x value, assuming that the above expression is relevant in the specified kinematic regime? (Take $n_f = 5$ for this estimate.)

Solution: This is just a numerical exercise to get a handle on the size of the enhancement do to the renormalization of the gluon distribution. Starting with

$$\frac{(xg(x, t_0))}{x} \simeq \frac{420}{99} \frac{(1-x)^7}{x},$$

we want to evaluate the enhancement of the small x behavior by the extra exponential factor above. With the chosen kinematic values we have

$$\ln \left(\frac{t}{t_0} \right) = \ln \left(\frac{\ln(100/0.1)}{\ln(5/0.1)} \right) = \ln(1.77) = 0.568.$$

For the choice $x = 0.01$, $\ln 1/x = 4.61$ and with $n_f = 5$ we obtain

$$\frac{48}{b_0} = \frac{48}{11 - 10/3} = 6.26$$

and

$$\exp \sqrt{\frac{48}{b_0} \ln \left(\frac{t}{t_0} \right) \ln \left(\frac{1}{x} \right)} = \exp \sqrt{6.26 \times 0.568 \times 4.61} = e^{\sqrt{16.4}} = 57.4.$$

This is a large enhancement indeed and it is even larger for smaller x values.

3. Let us focus briefly on the Drell-Yan process, the production of a virtual photon in hadron-hadron collisions via the annihilation of a quark and antiquark. The short distance (“hard”) process is the time reversed version of the electron-positron annihilation process. Thus $e^+e^- \rightarrow q\bar{q}$ becomes $q\bar{q} \rightarrow \mu^+\mu^-$, where the specific choice of lepton arises from the desire to employ a lepton pair that is “easily” detected. This cross section must then be convoluted with the appropriate parton distribution functions. In terms of the scaled virtual photon mass $\tau = Q^2/s$ and the photon rapidity $y = \frac{1}{2} \ln \left[(q_0 - q_z)/(q_0 + q_z) \right]$, the “scaling” or parton model version of the cross section looks like

$$s \frac{d\sigma}{d\sqrt{\tau} dy} = \frac{8\pi\alpha^2}{3\sqrt{\tau}} g(\sqrt{\tau}e^y, \sqrt{\tau}e^{-y}),$$

where the (LO) parton “luminosity” function has the form

$$g(x_a, x_b) = \frac{1}{3} \sum_f e_f^2 \left\{ q_f^a(x_a) \bar{q}_f^b(x_b) + \bar{q}_f^a(x_a) q_f^b(x_b) \right\}.$$

The label a, b correspond to the 2 incident hadrons. The explicit factor of $1/3$ is required because the conventional normalization of the $pdfs$ (q, \bar{q}) includes an implicit sum over colors. Here the quark-antiquark pair that annihilates must be of the same color. Thus the annihilation occurs for only $1/3$ of the possible pairs.

Contrary to the collider physics we have focused on in the Lectures, consider now the case of pion beams incident on a nuclear target, *i.e.*, composed of the canonical nucleon $N = (p+n)/2$. If we focus on large $\tau = Q^2/s$ so that we can safely assume that the interaction is dominated by the valence quarks (and antiquarks), determine the expected (and observed) value of the ratio

$$R_{DY} \equiv \frac{\sigma(\pi^+ N \rightarrow \mu^+ \mu^- X)}{\sigma(\pi^- N \rightarrow \mu^+ \mu^- X)}.$$

Solution: In the limit of valence quark dominance we have the nucleon represented by $N = (uud + udd)/2$, the positive pion by $\pi^+ = u\bar{d}$ and the negative pion by $\pi^- = d\bar{u}$. Thus the elementary Drell-Yan process at large τ is dominated by $u\bar{u}$ annihilation in the negative pion case and $d\bar{d}$ annihilation in the positive pion case (*i.e.*, the antiquark must come from the pion). Using the isospin symmetry of the strong interactions (*i.e.*, QCD) we expect the relevant parton distributions to be identical, *i.e.*,

$$\begin{aligned} \bar{d}_{\pi^+}(x) &= \bar{u}_{\pi^-}(x), \\ u_N(x) &= d_N(x). \end{aligned}$$

Thus the only difference in the two cases (*i.e.*, the 2 pion beams) will arise from the electric charges of the relevant valence quarks,

$$R_{DY}|_{\text{valence}} \equiv \frac{\sigma(\pi^+ N \rightarrow \mu^+ \mu^- X)}{\sigma(\pi^- N \rightarrow \mu^+ \mu^- X)} \simeq \frac{e_d^2}{e_u^2} = \frac{1/9}{4/9} = \frac{1}{4}.$$

This naïve prediction is in reasonable agreement with data.

4. In this exercise we want to become familiar with various features of collider kinematics. As noted in the Lecture the “real” rapidity and the pseudo-rapidity are defined by

$$\begin{aligned} \text{rapidity} = y_J &= 0.5 \ln \left(\frac{E + p_z}{E - p_z} \right) \\ \text{pseudo-rapidity} = \eta &= -\ln \left(\tan \left(\frac{\theta}{2} \right) \right) \end{aligned}$$

where the z direction is the direction of the beam.

a) Verify, as stated in the Lectures, that for any particle of mass M we can write

$$E = \sqrt{M^2 + p_T^2} \cosh y, \quad p_z = \sqrt{M^2 + p_T^2} \sinh y,$$

$$p_T^2 = p_x^2 + p_y^2.$$

Solution: The constraints to be satisfied are that $E = \sqrt{M^2 + p_z^2 + p_T^2} \geq M$ and that p_z covers the full range from $+\infty$ to $-\infty$. Clearly the state expression for the longitudinal momentum satisfies this last constraint for $+\infty < y < -\infty$. Then we need to check the form of the energy given p_z

$$E = \sqrt{M^2 + p_T^2 + p_z^2} = \sqrt{M^2 + p_T^2 + (M^2 + p_T^2) \sinh^2 y}$$

$$= \sqrt{M^2 + p_T^2} \cosh y \geq M.$$

b) Prove that $\tanh \eta = \cos \theta$ and thus that η is easy to measure.

Solution: We just use the definitions and calculate

$$\tanh \eta = \frac{\sinh \eta}{\cosh \eta} = \frac{e^\eta - e^{-\eta}}{e^\eta + e^{-\eta}} = \frac{\cot \frac{\theta}{2} - \tan \frac{\theta}{2}}{\cot \frac{\theta}{2} + \tan \frac{\theta}{2}} = \frac{\cos^2 \frac{\theta}{2} - \sin^2 \frac{\theta}{2}}{\cos^2 \frac{\theta}{2} + \sin^2 \frac{\theta}{2}}$$

$$= \cos \theta.$$

c) If particles are produced uniformly in longitudinal phase space with a differential distribution that looks like

$$dN = C \frac{dp_z}{E},$$

with C a constant, find the corresponding distribution in y , dN/dy .

Solution: We just need to use the definitions to find

$$\frac{dp_z}{E} = \frac{\sqrt{M^2 + p_T^2} (d \sinh y)}{\sqrt{M^2 + p_T^2} \cosh y} = dy.$$

Thus particles that are uniform in (relativistic) longitudinal phase are uniform in rapidity

$$E \frac{dN}{dp_z} = \frac{dN}{dy} = C.$$

This is a very good approximate description of the soft particles produced in the bulk of inelastic interactions (the ones not corresponding to some “hard” interaction). Their distribution is cut off rapidly (\sim exponentially) in p_T and is approximately flat in rapidity.

- d) Prove that the rapidity equals the pseudo rapidity, $\eta = y$, for a massless particle (and thus approximately for a relativistic particle, $E \gg M$).

Solution: For massless particles $E = \sqrt{M^2 + p^2} \rightarrow p$ and thus

$$\tanh y = \frac{p_z}{E} = \frac{p \cos \theta}{E} = \frac{p \cos \theta}{p} = \cos \theta = \tanh \eta$$

$$\Rightarrow y = \eta.$$

In the last step we used the result of part b), true for any E/M value.

- e) Prove that for a Lorentz transformation (boost) in the beam (z) direction, the rapidity, y , of every particle is shifted by a constant y_0 , which is related to the boost velocity. Recall the form of such a boost to a reference frame moving in the z direction with velocity u (with respect to the original frame and with $c = 1$)

$$\begin{aligned}
E' &= \gamma(E - \beta p_z), \\
p'_z &= \gamma(p_z - \beta E), \\
p'_x &= p_x, p'_y = p_y, \\
\beta &= u, \gamma = \frac{1}{\sqrt{1 - \beta^2}}.
\end{aligned}$$

Solution: We proceed by writing out the form of the boost in the notation of rapidities

$$\begin{aligned}
E' &= \sqrt{M^2 + p_T'^2} \cosh y' = \sqrt{M^2 + p_T^2} \cosh y' \\
&= \gamma(E - \beta p_z) = \gamma \sqrt{M^2 + p_T^2} (\cosh y - \beta \sinh y) \\
p'_z &= \sqrt{M^2 + p_T'^2} \sinh y' = \sqrt{M^2 + p_T^2} \sinh y' \\
&= \gamma \sqrt{M^2 + p_T^2} (\sinh y - \beta \cosh y).
\end{aligned}$$

Trying the Ansatz that $y' = y - y_0$ we immediately find the consistent solution that

$$\begin{aligned}
\cosh y' &= \cosh y - y_0 = \cosh y \cosh y_0 - \sinh y \sinh y_0 \\
&= \gamma(E - \beta p_z) = \gamma (\cosh y - \beta \sinh y) \\
\sinh y' &= \sinh y - y_0 = \sinh y \cosh y_0 - \cosh y \sinh y_0 \\
&= \gamma (\sinh y - \beta \cosh y) \\
\Rightarrow \cosh y_0 &= \gamma \geq 1, \sinh y_0 = \beta \gamma [-1 \leq \beta \leq 1] \\
\Rightarrow y_0 &= \sinh^{-1} \left(\frac{\beta}{\sqrt{1 - \beta^2}} \right) \\
\Rightarrow \cosh^2 y_0 - \sinh^2 y_0 &= \gamma^2 (1 - \beta^2) = \frac{1 - \beta^2}{1 + \beta^2} = 1.
\end{aligned}$$

Finally we note that this arithmetic depended only on the boost itself, not the original E , *i.e.*, not the original rapidity. Hence the rapidity of EVERY particle is shifted by

the same amount. A Lorentz transformation in the z direction is just a translation in rapidity (this is why invariant differential longitudinal phase space is proportional to dy)

- f) Consider a Z boson that is produced on-shell at the LHC in a $q\bar{q}$ annihilation process. The velocity of the Z boson is along the beam direction. What is the condition relating x_1 and x_2 , the momentum fractions of the quark and anti-quark. (Compare to the expressions in the previous exercise.)

Solution: The usual (parton model) kinematics (see the previous exercise) tell us that

$$E_Z = (x_1 + x_2) \frac{\sqrt{s}}{2}, \quad p_{z,Z} = (x_1 - x_2) \frac{\sqrt{s}}{2}, \quad p_{T,Z} = 0,$$

$$M_Z^2 = E_Z^2 - p_{z,Z}^2 = x_1 x_2 s \Rightarrow \tau \equiv \frac{M_Z^2}{s} = x_1 x_2.$$

Constructing the rapidity of the Z we find

$$y_Z = \frac{1}{2} \ln \left(\frac{E_Z + p_{z,Z}}{E_Z - p_{z,Z}} \right) = \frac{1}{2} \ln \left(\frac{x_1 + x_2 + x_1 - x_2}{x_1 + x_2 - (x_1 - x_2)} \right) = \frac{1}{2} \ln \frac{x_1}{x_2}$$

$$\Rightarrow \frac{x_1}{x_2} = e^{2y_Z} \Rightarrow x_1 = \sqrt{\frac{x_1}{x_2}} (x_1 x_2) = \sqrt{\tau} e^{y_Z} = \sqrt{\frac{M_Z^2}{s}} e^{y_Z}$$

$$x_2 = \sqrt{\frac{x_2}{x_1}} (x_1 x_2) = \sqrt{\tau} e^{-y_Z} = \sqrt{\frac{M_Z^2}{s}} e^{-y_Z}.$$

These relations between the momentum fractions (the x 's) and the rapidity (and the resonance mass) are just what was stated in the previous exercise.

5. Demonstrate that both the cone algorithm (without seeds) and the k_T algorithm are IRS at NLO in pQCD, *i.e.*, show that the “found” jet will have the same properties whether it contains a single parton or a pair of collinear partons with the same total momentum. Also show that the jet is unchanged by the emission of a (vanishingly)

soft gluon. This does not require a slick argument. The idea is just to give you the opportunity to think through what it takes to be IRS.

Solution: The point here is that the jet algorithms sum up the momentum of objects (partons, hadrons, calorimeter towers) that are nearby each other, in some sense. Either they are objects inside an angular cone of size R in the cone algorithm, or they are pair wise near each other in momentum space in the k_T algorithm. Thus a collinear pair of partons will always end up either in the same cone, or merged first in the k_T algorithm because they have $d_{ij} = 0$. For the case of a zero energy extra parton it may not end up in the same jet with the non-zero energy parton, either because it is outside of the cone or because it has the minimum d_i and is a separate jet, but this does not matter. It is still true that the algorithm will find the same non-zero energy jet (the other parton) as in the case of no extra emission. Thus the divergent parts of the real emission will contribute to the same jet (found by the algorithm) as the virtual emission contribution and the divergences will cancel. Hence the jet cross section defined by both the cone algorithm (without seeds) and the k_T algorithm are IRS.

6. Use the Snowmass definition of the iterative cone algorithm (i.e., E_T weighting instead of 4-vector addition) to show that the 2-parton phase space splits up as indicated in the figure in the Lecture. While this is really a 2-D problem in (y, ϕ) , the fact that there are only 2 partons, which effectively lie in a plane, means we can think of it as a 1-D problem, i.e., just the separation d in that plane.

Solution: The point here is to think about how the cone algorithm works in pQCD in order to understand some of the issues that have arisen. We can think in terms of a 2-D angular vector with components rapidity and azimuthal angle, $\vec{\Omega} \equiv (y, \phi)$. Now for 2 partons the angular separation is given by $\vec{\Omega}_{12} \equiv (y_1 - y_2, \phi_1 - \phi_2)$ with $|\vec{\Omega}_{12}| = d$. If we choose, for simplicity, to measure the rapidities and azimuthal angles from the geometric center of the cone, the location of the E_T weighted centroid is given by

$$\vec{\Omega}_C = \left(\frac{E_{T,1}y_1 + E_{T,2}y_2}{E_{T,1} + E_{T,2}}, \frac{E_{T,1}\phi_1 + E_{T,2}\phi_2}{E_{T,1} + E_{T,2}} \right) = \left(\frac{y_1 + zy_2}{1+z}, \frac{\phi_1 + z\phi_2}{1+z} \right),$$

where we have chosen $E_{T,1} \geq E_{T,2}$, $z = E_{T,2}/E_{T,1}$. Thus the condition that the E_T weighted centroid coincides with the geometric center of the cone is that

$$|\vec{\Omega}_C| = 0 \Rightarrow (y_1, \phi_1) = -z(y_2, \phi_2) \Rightarrow |(y_1, \phi_1)| \leq |(y_2, \phi_2)|.$$

Thus, as expected, the more energetic parton must be closer to the center of the cone than the less energetic one. So the condition that both partons are inside a cone of radius R centered at the E_T weighted centroid is that $|\vec{\Omega}_{C2}| \leq R$. Thus the effective (1-D) boundary for finding a single cone with both partons in side is given by

$$\vec{\Omega}_{12} = (y_1 - y_2, \phi_1 - \phi_2) = (1+z)(-y_2, -\phi_2),$$

$$d = |\vec{\Omega}_{12}| = (1+z)|\vec{\Omega}_{C2}| \leq (1+z)R.$$

This is what is meant to be illustrated in the figure.

