Strong-coupling lattice QCD at high density

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I. Outline of the talk

II. **Motivation** and general strategy.

III. H_{eff} - Derivation and structure.

IV. **How** do you **insert density** to the lattice ?

V. **Analogy** to the Heisenberg anti-ferromagnet.

VI. **NLSM** for the problem.

VII. **Classical** ground state of the NLSM.

IIX-IX. **“Order from disorder”** - the role of quantum fluctuations.

X. **SSB’s of chiral symmetry:** for different densities.

XI. **Future** research.

XII. **Summary.**

II. Motivation

Recent years – great interest in QCD at nonzero density, in particular idea of color-superconductivity (CSC) where $\langle qq \rangle \neq 0$.

CSC analysis – based on QCD inspired models with weak coupling – Need a non perturbative analysis but,

Standard Monte Carlo simulations within lattice QCD **fail** at $\mu \neq 0$ due to the sign problem:

$$P \sim e^{-S(A)} \sim e^{-S_{\mathrm{YM}}(A)} \det D(A) \notin \Re \quad \text{for} \quad \mu \neq 0.$$

What can strong coupling lattice QCD tell us about finite density?

We use Hamiltonian lattice QCD at $\boldsymbol{g \gg 1}$, far from $a \rightarrow 0$:

- Treat it as an effective Hamiltonian for QCD at large distances.
- $U(N_f) \times U(N_f)$ symmetry, confinement, and the ordinary SSB of chiral symmetry → a model for QCD at large distances.
- Fermions are naive (nearest-neighbor) + next-nearest-neighbor → doubling in weak-coupling, but we work at strong coupling, where we don't have femrion excitations.

III. Hamiltonian QCD at strong coupling

$\boxed{H = H_{\mathrm{E}} + H_{\mathrm{F}} + H_{\mathrm{U}}}$

$\mathbf{n}$–site, $\mu = x, y, z$

$$\begin{aligned}
H_{\mathrm{E}} &= \frac{g^2}{2}\sum_{\mathbf{n}\mu} E^2_{\mathbf{n}\mu}, \\
H_{\mathrm{F}} &= -i \sum_{\substack{\mathbf{n}\mu,\\ l>0}} \psi^\dagger_{\mathbf{n}} \alpha_\mu D_l \left(\prod U_{\mathbf{n}+k\hat{\mu},\mu}\right) \psi_{\mathbf{n}+l\mu} + h.c. \\
&= -i \sum_{\mathbf{n}\mu} \psi^\dagger_{\mathbf{n}} \alpha_\mu \left\{ D_1 U_{\mathbf{n},\mu} \psi_{\mathbf{n}+\mu} + D_2 U_{\mathbf{n},\mu} U_{\mathbf{n}+\hat{\mu},\mu} \psi_{\mathbf{n}+2\mu} \right\} + h.c. \\
H_{\mathrm{U}} &= \frac{1}{2g^2} \sum_{\mathrm{P}} \mathrm{Tr}\left[1 - U_{\mathrm{P}}\right].
\end{aligned}$$

$\Psi^\dagger$ creates quarks

$U_{\mathbf{n}\hat{\mu}}$ creates flux

E^2 measures flux

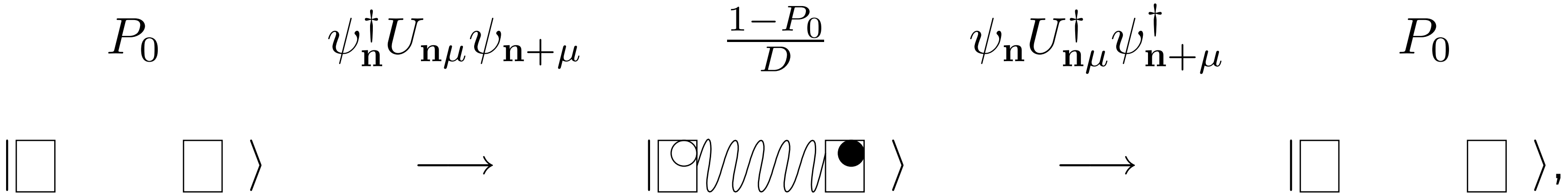
At strong coupling $g \gg 1$:

- $H \simeq H_{\mathrm{E}}$. Since $E^2_{\mathbf{n}\mu}$ measures flux on link $\mathbf{n}\mu$:
- unperturbed ground state: $|0\rangle = \prod_{\mathbf{n}\mu} |E^2_{\mathbf{n}\mu} = 0\rangle \otimes |\psi\rangle$, is very degenerate.

Degenerate perturbation theory in $H_{\rm F}$:

$$H_{\rm eff} = P_0 H_F \frac{1-P_0}{D} H_F P_0 \quad + \quad O(H_{\rm F}^3) \equiv H_{\rm eff}^{(2)} + O(H_{\rm F}^3)$$

Pictorially ($l = 1$):



$$P_0 \qquad \psi^\dagger_{\bf n} U_{{\bf n}\mu} \psi_{{\bf n}+\mu} \qquad \frac{1-P_0}{D} \qquad \psi_{\bf n} U^\dagger_{{\bf n}\mu} \psi^\dagger_{{\bf n}+\mu} \qquad P_0$$

So the fermion number on a site does not change, and the color singlet effective Hamiltonian is

$$H_{\rm eff}^{(2)} \sim \left(\psi^{\dagger a}_{\alpha{\bf n}} \psi^a_{\beta{\bf n}}\right) \left(\psi^{\dagger b}_{{\bf n}+\mu,\delta} \psi^b_{{\bf n}+\mu,\gamma}\right), \qquad \left[H_{\rm eff}^{(2)}, \Psi^\dagger_{\bf n} \Psi_{\bf n}\right] = 0,$$

and changes Dirac and flavor indices without changing fermion number, which means:

$$\boxed{H_{\rm eff}^{(2)} \text{ describes physics with } \underline{\text{fixed}} \text{ baryon background.}}$$

$H_{\text{eff}}^{(2)}$–the Heisenberg Anti-Ferromagnet

$$H_{\text{eff}}^{(2)} = \sum_{\substack{\mathbf{n}\mu \\ l=1,2}} J_l \left(s_\eta^\mu\right)^{l+1} Q_{\mathbf{n}}^\eta Q_{\mathbf{n}+l\hat{\mu}}^\eta .$$

- $\boldsymbol{J_{1,2}}$ depends on the fermion kernel $(D_{1,2})$ and obey $0 < J_2 \ll J_1$.
- $\boldsymbol{Q^\eta}_{\mathbf{n}}$ are fermion bilinears:

$$Q_{\mathbf{n}}^\eta = \sum_{a=1}^{N_c} \sum_{\alpha,\beta=1}^{4N_f} \Psi_{\mathbf{n}}^{\dagger\alpha a} M_{\alpha\beta}^\eta \Psi_{\beta a\mathbf{n}} - \frac{N_c}{2} \mathrm{Tr}\, M^\eta .$$

- **The set** $\boldsymbol{M^\eta} = \Gamma^A \otimes \lambda^f$ are the generators of $U(4N_f)$, acting on Dirac-flavor indices:

The “spin” operators $\boldsymbol{Q_{\mathbf{n}}^\eta}$ generate a $U(4N_f)$ algebra.

$$\boxed{H_{\mathrm{eff}}^{(2)} = \sum_{\substack{\mathbf{n}\mu \\ l=1,2}} J_l \left(s_\eta^\mu\right)^{l+1} Q_{\mathbf{n}}^\eta Q_{\mathbf{n}+l\hat{\mu}}^\eta .}$$

- $\boldsymbol{s_\eta^\mu}$ is a sign factor:

$l = 1$ (NN):

$$H_{NN} \sim \sum_\eta Q_{\mathbf{n}}^\eta Q_{\mathbf{n}+\mu}^\eta \equiv \vec{Q}_{\mathbf{n}} \vec{Q}_{\mathbf{n}+\mu} \to H_{\mathrm{eff}}^{(2)}$$ — $U(4N_f)$ NN Heisenberg AFM.

$l = 2$ (NNN):

$$H_{NNN} \sim Q_{\mathbf{n}}^1 Q_{\mathbf{n}+2x}^1 - Q_{\mathbf{n}}^2 Q_{\mathbf{n}+2y}^2 + \ldots \to$$ Breaks $U(4N_f)$ to $U(N_f)_L \times U(N_f)_R$.

Same as in underlying Hamiltonian:

"Spin-diagonalize" the $4N_f$–component spinor $\boxed{\psi_{\mathbf{n}} = T_{\mathbf{n}} \chi_{\mathbf{n}}}$ to eliminate α_μ from NN

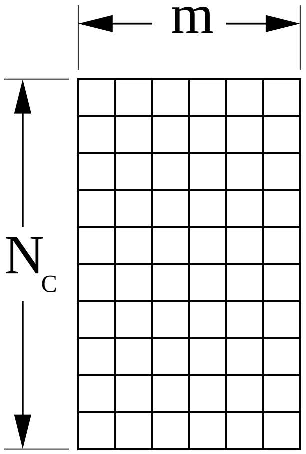
$$\psi_{\mathbf{n}}^\dagger \alpha_\mu \psi_{\mathbf{n}+\mu} = \chi_{\mathbf{n}}^\dagger T_{\mathbf{n}}^\dagger \alpha_\mu T_{\mathbf{n}+\mu} \chi_{\mathbf{n}+\mu} \propto \chi_{\mathbf{n}}^\dagger \cdot 1 \cdot \chi_{\mathbf{n}+\mu} \to$$ NN is $U(4N_f)$ symmetric.

Spin-diagonalization does not remove α_μ from NNN terms, so only by including these you really have $U(N_f)_L \times U(N_f)_R$.

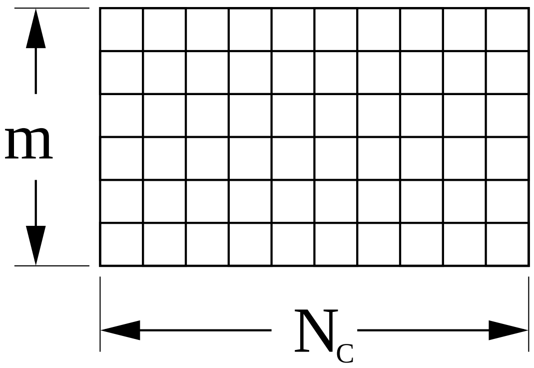
IV. What about density?

Gauss' law → all sites must be color singlets so the number of fermions per site must be a multiple of N_c, in particular a quantum state on a site with $n = m \cdot N_c$ fermions must have

Color indices



Dirac-flavor indices



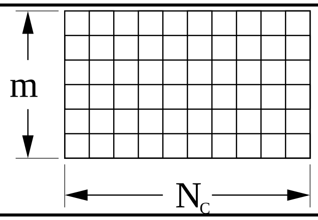
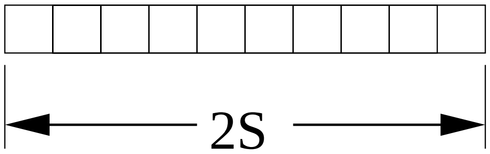
So the fermionic occupancy on a site determines the $U(4N_f)$ representation of Q^η on that site.

[Representation distribution] $\equiv$ [B distribution], with

$$\boxed{B_{\mathbf{n}} = m_{\mathbf{n}} - 2N_f} \quad .$$

Since $[H_{\text{eff}}, B_{\mathbf{n}}] = 0$ Hilbert space factors according to $B_{\mathbf{n}}$; solve H_{eff} in each factor.

V. A pause – analogy to the Heisenberg AFM

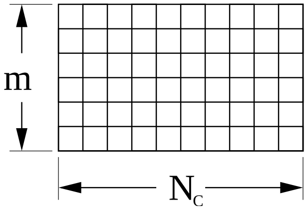
$H^{(2)}_{\text{eff}}$	Heisenberg AFM
The problem	
Q^η generate $U(4N_f)$ with $\eta \in [1,(4N_f)^2]$	S^i generate $SU(2)$ with $i=x,y,z$
	
$H \sim \sum_{\mathbf{n}\mu}\sum_\eta Q^\eta_{\mathbf{n}} Q^\eta_{\mathbf{n}+\mu}$	$H \sim \sum_{\mathbf{n}\mu} \vec{S}_{\mathbf{n}} \cdot \vec{S}_{\mathbf{n}+\mu}$
NNN break $U(4N_f)$ to $U(N_f)\times U(N_f)$	e.g. NNN $= S^z_{\mathbf{n}} S^z_{\mathbf{n}+2\mu}$ breaks $SU(2)$ to $U(1)$
Generalized "spin" coherent states $\sigma \in$?	Spin coherent state: Write Z as a path integral for NLSM $\sigma \in SU(2)/U(1) \simeq S^2$
large N_c	large S limit

VI. Spin coherent states (σ model) representation

On site $\mathbf{n}$, with baryon number $B_{\mathbf{n}} = m - 2N_f$, define "spin" coherent states

$$|q\rangle \equiv \exp\left(\sum_{\alpha=1}^{m}\sum_{\beta=m+1}^{4N_f}\left(q_\beta^{*\alpha}\Psi^{\dagger\beta a}\Psi_{\alpha a} - q_\beta^{\alpha}\Psi^{\dagger\alpha a}\Psi_{\beta a}\right)\right)|0\rangle.$$

$|0\rangle$ – highest weight state in



Write a path integral for the partition function → NLSM:

- σ fields are $4N_f \times 4N_f$ matrices that are both unitary and hermitian:

$$\sigma_{\mathbf{n}} = U_{\mathbf{n}}\Lambda U_{\mathbf{n}}^\dagger \propto \langle q|\vec{Q}_{\mathbf{n}}\cdot\vec{M}|q\rangle \quad \text{with} \quad U_{\mathbf{n}} \in U(4N_f), \text{ and } \Lambda = \begin{pmatrix} 1_m & 0 \\ 0 & -1_{4N_f-m} \end{pmatrix}$$

- This means that $\boldsymbol{\sigma_{\mathbf{n}} \in U(4N_f)/[U(m_{\mathbf{n}}) \times U(4N_f - m_{\mathbf{n}})]}$, depends on $B_{\mathbf{n}}$.

- The action is

$$S = \frac{\mathbf{N_c}}{\mathbf{2}} \int_0^\beta d\tau \, \mathrm{Tr} \left(\sum_{\mathbf{n}} -\Lambda U_{\mathbf{n}}^\dagger \partial_\tau U_{\mathbf{n}} + \mathcal{H}_{\mathrm{eff}}^{(2)} \right) .$$

- First term is a Berry phase (it is imaginary), comes from the overlap $\langle q(\tau) | q(\tau+\epsilon) \rangle$, and is the usual $\sim i\pi\partial_t\Phi$ that gives S from H.

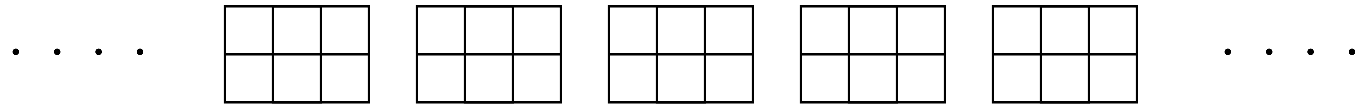
- The second term is the quantum $\frac{N_c}{4} \times H_{\mathrm{eff}}^{(2)}$ with the operators $\vec{Q} \cdot \vec{M}$ replaced by σ.

- For NN+NNN interactions you get

$$\mathcal{H}_{\mathrm{eff}}^{(2)} = \frac{N_c}{4} \sum_{\mathbf{n},\mu} J_1 \left[\sigma_{\mathbf{n}} \sigma_{\mathbf{n}+\hat{\mu}} \right] + J_2 \left[\sigma_{\mathbf{n}} \alpha_\mu \sigma_{\mathbf{n}+2\hat{\mu}} \alpha_\mu \right] \sim O(1/(g^2 N_c))$$

VII. Large N_c and classical vacuum

Restrict to uniform density $\boldsymbol{B_{\mathbf{n}} = B \geq 0}$,



Restrict to $\boldsymbol{J_2 = 0}$, so symmetry is the too large $\boldsymbol{U(4N_f)}$.

At large N_c Minimize action:

- $\partial_\tau \to 0$.
- Minimize $\mathcal{H}^{(2)}_{\mathrm{eff}}$ – minimize the one link problem:

$$E_{\mathrm{link}} = J_1 \mathrm{Tr}\, \sigma_1 \sigma_2 \geq -J_1 \cdot 4N_f \quad (\mathrm{because}\, \sigma_1 \cdot \sigma_2 \in U(4N_f))$$

$B = 0$, a warm up

- $m = 2N_f$, so the σ fields are unitary rotations of $\quad \Lambda = \begin{pmatrix} 1_{2N_f} & 0 \\ 0 & -1_{2N_f} \end{pmatrix}.$
- without loss of generality, take $\sigma_1 = \Lambda$.
- Choose σ_2 as a rotation that switches the blocks of Λ: $-1_{2N_f} \to +1_{2N_f}$.

$$J_1 \mathrm{Tr} \begin{pmatrix} 1_{2N_f} & 0 \\ 0 & -1_{2N_f} \end{pmatrix} \begin{pmatrix} -1_{2N_f} & 0 \\ 0 & +1_{2N_f} \end{pmatrix} = -J_1 \cdot 4N_f.$$

- g.s. of link is $\sigma_2 = -\sigma_1$, ↗↙ .
- g.s. of lattice is $\sigma_{\mathbf{n}} = \begin{cases} \sigma_1 & \text{even} \\ -\sigma_1 & \text{odd} \end{cases}$, an AFM.
- SSB is

$$\boldsymbol{U(4N_f) \to U(2N_f) \times U(2N_f).}$$

So what happens for $B > 0$

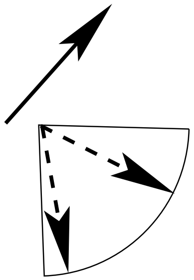
- $m > 2N_f$, and $m > 4N_f - m$, so the σ fields are unitary rotations of

$$\Lambda = \begin{pmatrix} 1_m & 0 \\ 0 & -1_{4N_f-m} \end{pmatrix}.$$

- again, without loss of generality, take $\sigma_1 = \Lambda$.
- σ_2: as a rotation that switches lower block -1_{4N_f-m} with <u>**any**</u> part of upper block → **accidental degeneracy**:

$$\begin{aligned} \sigma_2 &= \left(\begin{array}{c|c} \sigma_m & 0 \\ \hline 0 & 1_{4N_f-m} \end{array}\right), \\ \sigma_m &\equiv u_m \begin{pmatrix} 1_{2m-4N_f} & 0 \\ 0 & -1_{4N_f-m} \end{pmatrix} u_m^\dagger, \qquad u_m \in U(m). \end{aligned}$$

- g.s. of link is: $\sigma_2 \in \left(\frac{U(m)}{U(2m-4N_f)\times U(4N_f-m)}\right)_{\sigma_1}$

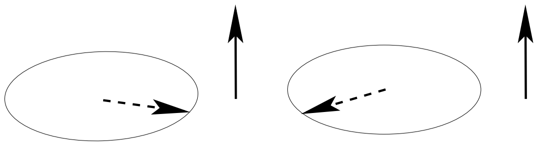


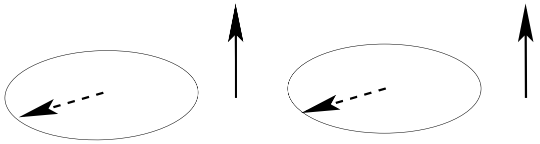
- Take all $\sigma_{\text{even}} = \sigma_1$, **each** σ_{odd} is free to move in a submanifold relative to σ_1.

Classical g.s. of the $\boldsymbol{B} \neq \mathbf{0}$ lattice is locally degenerate.

- Local degeneracy → have fields ("zero modes") that change the classical g.s. but have $\epsilon_{\mathbf{k}} = 0,\ \forall \mathbf{k}$.

- Similar to **"order from disorder"** in condensed matter.
  Example – Double exchange model:
  - Classically, between two adjacent spins – angle θ, e.g. $\theta = 90^0$:

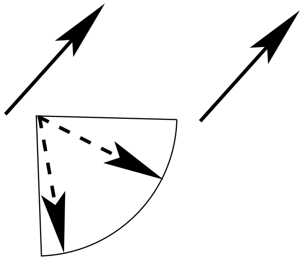


  - Also have zero modes.
  - Why **"order from disorder" ?** – fluctuations choose g.s. → remove degeneracy, and pick canted state:  → zero modes get energy.

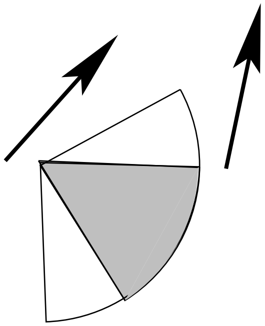
IIX. Quantum fluctuations remove local degeneracy

Why do the even sites want to align ?

Aligned:



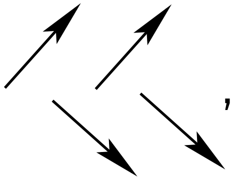
Not aligned:



What about odd sites? — also aligned, same reasoning.

System prefers FM alignment of NNN, have “canted” g.s.

$$\sigma_{\mathbf{n}} = \begin{cases} \Lambda & \text{even} \\ \sigma_2 & \text{odd} \end{cases}$$



, Can take $\sigma_2 = \begin{pmatrix} 1_{2m-4N_f} & 0 & 0 \\ 0 & -1_{4N_f-m} & 0 \\ 0 & 0 & 1_{4N_f-m} \end{pmatrix}$,

These vevs lead to $\boldsymbol{U(4N_f) \to U(2m-4N_f) \times U(4N_f-m) \times U(4N_f-m)}$

$^*\Lambda = \operatorname{diag}\left(1_m, -1_{4N_f-m}\right) = \operatorname{diag}\left(1_{2m-4N_f}, 1_{4N_f-m}, -1_{4N_f-m}\right)$

IX. Stability of ground state - how fluctuations remove "zero modes"

Write $\boldsymbol{\sigma}_{\mathbf{n}}$ function of manifold coordinates $\pi_{\mathbf{n}}$ and $\chi_{\mathbf{n}}$.

- $\pi_{\mathbf{n}}$ – fluctuations **outside** of degenerate submanifold.

- $\chi_{\mathbf{n}}$ – fluctuations **within** degenerate submanifold.

Expand the action around "canted" g.s., in momentum space, after rescaling fields by $1/\sqrt{N_c}$:

$$S = S_0^{\text{classical}} + \pi^\dagger G_\pi^{-1}\pi + \chi^\dagger G_\chi^{-1}\chi + O(\frac{1}{\sqrt{N_c}}),$$

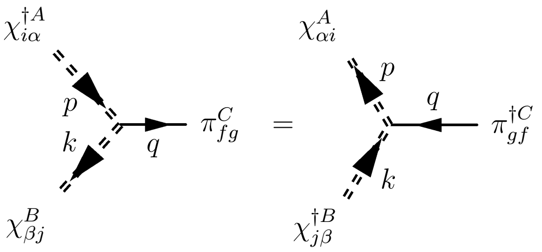
Bare propagators:

$$G_\pi \sim \frac{J_1}{\omega^2 + cJ_1^2|\mathbf{k}|^2}, \qquad |\mathbf{k}| \ll 1; \quad - \text{ ordinary GBs/AFM spin waves.}$$

$$G_\chi \sim \frac{1}{i\omega}, \qquad \forall \mathbf{k}; \quad - \text{ ``Zero modes''.}$$

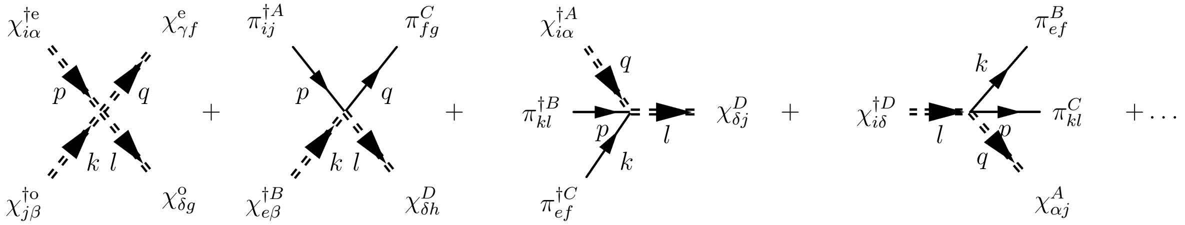
Self-energy – to $O(1/N_c)$

Vertices of S: $\quad \dfrac{1}{\sqrt{N_c}}\Bigg($



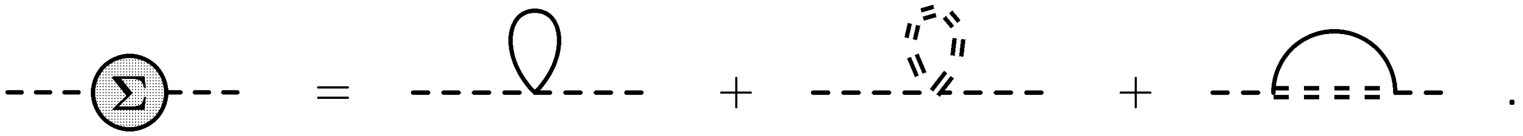
$$= -2Jd\sqrt{\tfrac{2}{N_sN_c\beta}}\,\delta_{\alpha\beta}\delta_{jf}\delta_{gi}\left[\delta_{A\mathrm{o}}\delta_{B\mathrm{e}}\delta_{C\mathrm{o}} - (\mathrm{e}\leftrightarrow\mathrm{o})\right]\gamma_{\boldsymbol{k}}.\Bigg) +$$

$+\dfrac{1}{N_c}\Bigg($



$$= -2Jd\tfrac{2}{N_sN_c\beta}\delta_{if}\delta_{jg}\delta_{\gamma\beta}\delta_{\alpha\delta}\gamma_{\boldsymbol{p}-\boldsymbol{q}} + 2Jd\tfrac{2}{N_sN_c\beta}\delta_{gi}\delta_{je}\delta_{\beta\delta}\delta_{hf}\left[\delta_{A\mathrm{e}}\delta_{C\mathrm{e}}\delta_{B\mathrm{o}}\delta_{D\mathrm{o}} + (\mathrm{e}\leftrightarrow\mathrm{o})\right]\gamma_{\boldsymbol{p}-\boldsymbol{q}} - Jd\tfrac{2}{N_sN_c\beta}\delta_{\alpha\delta}\delta_{jk}\delta_{le}\delta_{if}\left[\delta_{A\mathrm{e}}\delta_{B\mathrm{e}}\delta_{C\mathrm{o}}\delta_{D\mathrm{e}} + (\mathrm{e}\leftrightarrow\mathrm{o})\right]\gamma_{\boldsymbol{p}} + \ldots.\Bigg)$$

Replace $G_\chi^{-1} = G_{\chi,\mathrm{bare}}^{-1} - \Sigma$, and get Schwinger-Dyson equation for Σ:



Schwinger-Dyson Eq leads to 2 coupled non-linear *integral* equations for $\boldsymbol{\Sigma(\omega, \mathrm{k})}$.

To $\boldsymbol{O(1/N_c)}$ equations become: $\boldsymbol{\tanh\theta_{\mathrm{k}} = \dfrac{c_B \int_{\mathrm{BZ}} \left(\frac{dq}{4\pi}\right)^d I_2(\mathrm{q},\mathrm{k}) \sinh\theta_{\mathrm{q}}}{\int_{\mathrm{BZ}} \left(\frac{dq}{4\pi}\right)^d I_1(\mathrm{q},\mathrm{k}) \cosh\theta_{\mathrm{q}} - \eta(\mathrm{k})}.}$

Solution: Get $\boldsymbol{\Sigma(\mathrm{k} \ll 1) \sim \frac{J_1}{N_c}\mathrm{k}^2}$, $\longrightarrow$ **Quadratic** dispersion for χ, typical to **FM**.

To conclude NN results

- π fields: Type I GBs/AFM spin waves: $\epsilon^I_{\mathrm{k}} \sim J_1|\mathbf{k}|$, $\boldsymbol{n_I = 2(4N_f - m)^2}$.
- χ fields: Type II GBs/FM magnons: $\epsilon^{II}_{\mathrm{k}} \sim \frac{J_1}{N_c}|\mathbf{k}|^2$, $\boldsymbol{n_{II} = 2(4N_f - m)(2m - 4N_f)}$.
- Nielsen-Chadha counting rule is obeyed: $n_I + 2n_{II} = n_G$, with

$$n_G = \dim\left[\frac{U(4N_f)}{U(2m - 4N_f) \times U(4N_f - m) \times U(4N_f - m)}\right]$$

X. What about the NNN interactions

Taking $J_2 > 0$ is important since it explicitly breaks $U(4N_f) \to U(N_f) \times U(N_f)$.

We take $J_2 \ll J_1$ $\to$ Perturb NN vacuum with NNN.

Vacuum alignment: Minimization of NNN interactions removes some of the global degeneracy of the g.s.

For example $B = 0$:

- NN: $\sigma_{\text{even}} = -\sigma_{\text{odd}} = U \begin{pmatrix} 1_{2N_f} & 0 \\ 0 & -1_{2N_f} \end{pmatrix} U^\dagger, \quad U \in U(4N_f)$
- NNN: fix U to minimize NNN interaction $E_{\text{NNN}} = J_2 \sum_\mu \sum_{a=\substack{\text{even}\\\text{odd}}} \text{Tr}\left[\sigma_a \alpha_\mu \sigma_a \alpha_\mu\right]$

$U = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$, and $\sigma_{\text{even,odd}} = \pm \begin{pmatrix} 0 & -1_{2N_f} \\ -1_{2N_f} & 0 \end{pmatrix}$ minimizes E_{NNN}.

This spontaneously breaks (basis $=\gamma_5$ diagonal)

$$U(N_f) \times U(N_f) \to U(N_f)_V.$$

N_f	$\lvert B \rvert$	Unbroken symmetry	Broken charges
1	0	$-$	1
	1	$-$	1
	2	$U(1)_A$	0
2	0	$SU(2)_V$	4
	1	$U(1)_{I_3}$	6
	2	$SU(2)_V$	4
	3	$U(1)_{I_3}$	6
	4	$SU(2)_L \times SU(2)_R \times U(1)_A$	0
3	0	$SU(3)_V$	9
	1	$U(1)_Y \times SU(2)_V$	13
	2	$U(1)_Y$	16
	3	$SU(3)_V$	9
	4	$U(1)_{I_3} \times U(1)_Y$	15
	5	$U(1)_{I_3} \times U(1)_Y \times U(1)_{A'}$	14
	6	$SU(3)_L \times SU(3)_R \times U(1)_A$	0

- Some type II get a mass $M \sim J_2$.
- Some type II become type I, with $\epsilon_{\mathbf{k}} \sim \sqrt{J_2 J_1 / N_c}\sqrt{c_z^2 k_z^2 + c_\perp^2 k_\perp^2}$.
- Some remain type II, but with $\epsilon_{\mathbf{k}} \sim J_1/N_c(c_z^2 k_z^2 + c_\perp^2 k_\perp^2)$.
- The rest do not change.

XI. What next ?

Go to higher orders of effective Hamiltonian:

- At $N_c = 3$, the 3rd order in perturbation theory moves three quarks from site to site.
- This makes the baryons dynamic,

$$H^{(3)} \sim \sum_{\mathbf{n},\mathbf{m}} K_{\mathbf{n},\mathbf{m}} b^{\dagger I}_{\mathbf{n}} b^{I}_{\mathbf{m}} + h.c.; \qquad b^I = \epsilon_{abc} \Psi^a_{I_1} \Psi^b_{I_2} \Psi^c_{I_3}; \qquad \{b^I, b^{I'\dagger}\} \neq \delta_{II'}$$

- And the Hamiltonian similar to the t–J model of condensed matter,

$$H_{t-J} = -t \sum_{\substack{\langle ij \rangle \\ s}} c^\dagger_{is} c_{js} + J \sum_{\langle ij \rangle} \left(\vec{S}_i \cdot \vec{S}_j - \frac{n_i n_j}{4} \right)$$

Fix μ instead of $B_{\mathbf{n}}$ and calculate in the NLσm. Now have two fields: $B_{\mathbf{n}} \in [-2N_f, 2N_f]$ and $\sigma_{\mathbf{n}} \in \mathcal{M}_{\mathbf{n}}(B_{\mathbf{n}}) \subset U(4N_f)$.

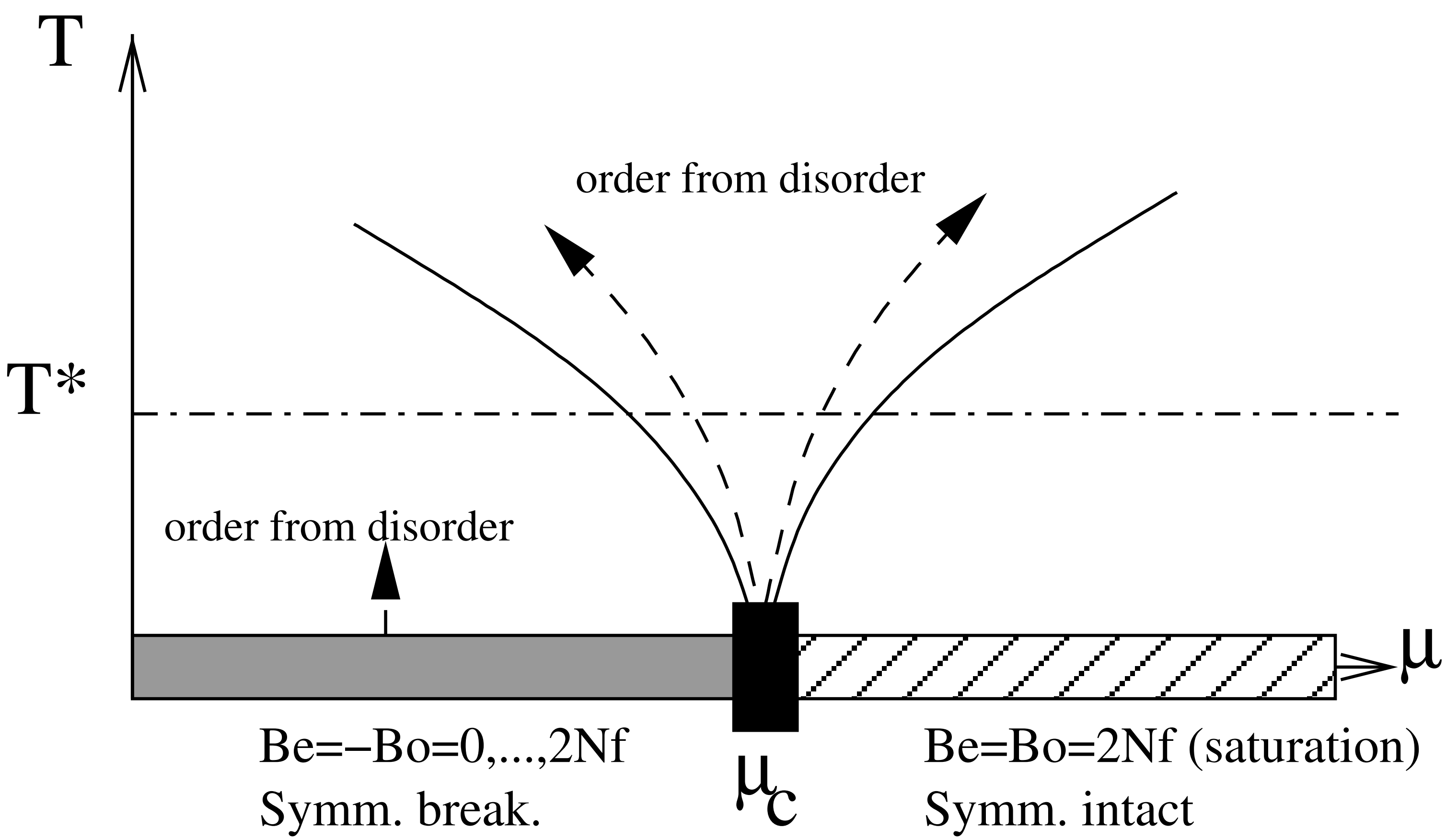
$$Z_{\rm GC}(\mu) = \sum_{\{B_{\mathbf{n}}\}} e^{\beta\mu N_c \sum_{\mathbf{n}} B_{\mathbf{n}}} Z_{\rm C}(B_{\mathbf{n}}) \qquad ; \qquad \int_{\mathcal{M}_{\mathbf{n}}} d\sigma_{\mathbf{n}} = \lambda_{\mathbf{n}} \int_{U(4N_f)} dU$$

$$Z_{GC} = \sum_{\{B_{\mathbf{n}}\}} \int_{U(4N_f)} DU \prod_{\mathbf{n}} \lambda_{\mathbf{n}}(B_{\mathbf{n}}) \times \exp\left\{ \frac{N_c}{2} \int_0^\beta d\tau \, {\rm Tr} \left(\sum_{\mathbf{n}} \Lambda U_{\mathbf{n}}^\dagger \left(\partial_\tau - \mu\right) U_{\mathbf{n}} + \mathcal{H}_{\rm eff}^{(2)} \right) \right\}$$

Again because of $S_{\rm kinetic} \in {\rm Im}$, so either take $N_c \to \infty$ or $\beta \to 0$ to get rid of ∂_τ, so at $T \to \infty$:

$$Z_{\rm GC} \to Z_{\rm classical} = \sum_{\{B_{\mathbf{n}}\}} \int_{U(4N_f)} DU \prod_{\mathbf{n}} \left[\lambda_{\mathbf{n}}(B_{\mathbf{n}}) e^{N_c \mu B_{\mathbf{n}}}\right] \times \exp\left\{ \frac{\beta N_c}{2} {\rm Tr}\, \mathcal{H}_{\rm eff}^{(2)} \right\}$$

Which gives (for the <u>classical model</u>)



XII. Summary

I. $g \gg 1 \to \underline{\boldsymbol{H}_{\text{eff}}^{(\mathbf{2})}}$ - Physics of fixed $B_{\mathbf{n}}$.

II. $H_{\text{eff}}^{(2)}$ is AFM → write a NLSM and solve at large N_c.

III. **Classical g.s.** of the $B > 0$ case is very degenerate, have zero modes.

IV. **Quantum fluctuations** remove degeneracy and pick FM alignment of NNN — "order from disorder".

V. **Quantum fluctuations** give zero modes $\epsilon_{\mathbf{k}} \sim |\mathbf{k}|^2$.

VI. **NNN** interactions perturb NN g.s. and found a variety of SSB, including breakdown of lattice rotations.

VII. As a result some of the GBs get anisotropic dispersion relations.