The deconfinement transition in $SU(N)$ lattice gauge theories at large-N

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- Focus on $4D$, recent $3D$ at large-N see **Liddle and Teper '05-'06, Holland '05.**
- Recent years activity, not older reduced models (but see **Das '87**).

Outline of the talk

I. **Large-N from the lattice** :

- Background and lattice techniques.

II. **$T \simeq T_d$ - nature of the transition** :

- Order of the transition.
- Deconfinement vs. Hagedorn.

III. **$T > T_d$ - Properties of the deconfined phase** :

- Bulk thermodynamics.
- Debye masses and spatial string tension.
- Domain wall tension.
- Topology.
- For χS-restoration see **Narayanan's talk.**

IV. **Summary**

I. Large N from the lattice

't Hooft '72 : For $SU(N)$, take $N \longrightarrow \infty$ with $g^2 N$ fixed → expansion in $1/N$.

Witten '79 : $N = \infty$ "planar" is a simpler theory:

- Free mesons and glueballs (no decays, mixings or scatterings).
- Classical, soliton-like, baryons ($m_B \to \infty$).

Since N^2 gluons $\gg$ N quarks : $\mathrm{QCD}_\infty = (\text{quenched QCD})_\infty$

$SU(\infty)$ still unsolved→ **lattice :**

Define $SU(N)$ on a 4D torus of $L_s^3 \times L_t$, S(Wilson), $\beta = \dfrac{2N}{g^2} \sim N^2$.

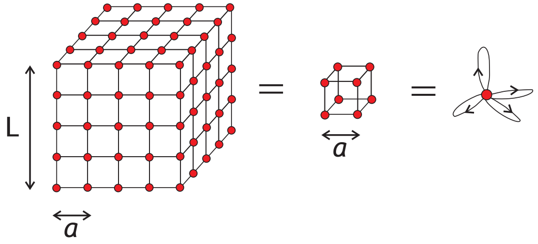
Standard route of large-N from the lattice : Do $SU(2, 3, 4, 6, 8, \ldots)$

$$\langle O(N)\rangle = \langle O(\infty)\rangle + \frac{a}{N^2} + \frac{b}{N^4} \ldots \Leftrightarrow \text{get limit + corrections.}$$

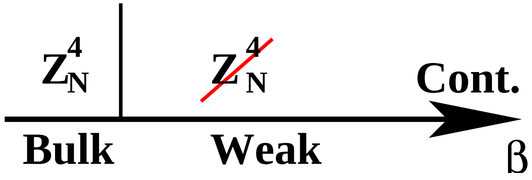
Teper and collaborators '98-'06, Del Debbio, Panagoupolous and Vicari '02,'06, de Forcrand, Lucini and co. '04,'05

Less standard route of large-N from the lattice : reduction

Eguchi-Kawai '82:

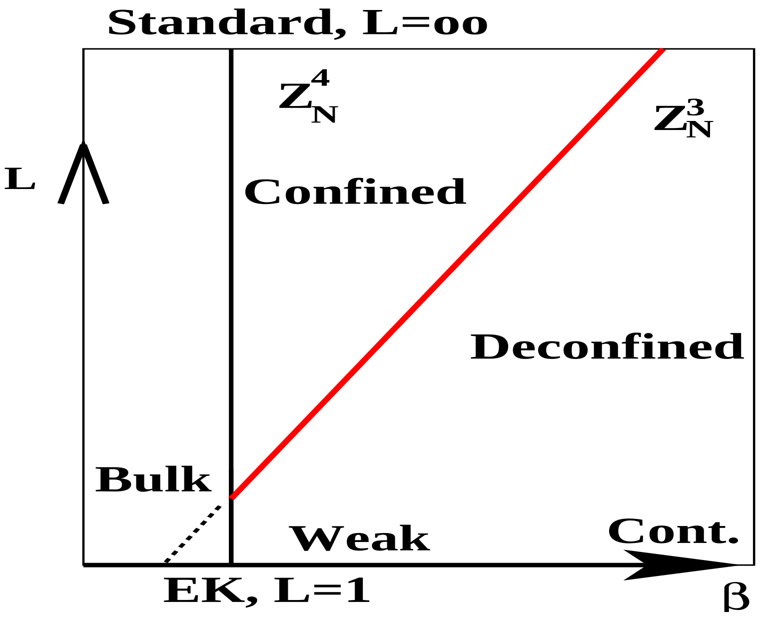


$+\ \mathcal{O}(1/N^2)$ but



- Fails in continuum **Bhanot, Heller and Neuberger '82** → Twisted,Quenched EK ('80) (**Das and Kogut, Fabricius and Haan, Gocksch and Neri, Gonzalez-Arroyo and Okawa, Klinkhamer, Neuberger, Rossi**).

A variant of EK reduction : partially reduce $\infty^4 \rightarrow L^4 > 1^4$ **Neuberger et al '02-'06**.



- **No $1/V$ at $N=\infty$ for $L > L_{\text{transition}}$**

II. $T \simeq T_d$ - nature of the transition

The Ultimate temperature idea by **Hagedorn ‘65** - pre-dated QCD

pp particle multiplicities → hadrons consistent only $T < T_H \simeq 158$ MeV.

Interpret T_H as 2nd-order transition of quark liberation by **Cabibo and Parisi ‘75**.

Heuristic description for Hagedorn/deconfinement given by **Banks and Rabinovici ‘79**:

Confinement means

$$\left.\begin{array}{l} E(l) = \sigma l \\ \rho(l) = \exp\left(+cl\right) \end{array}\right\} \rightarrow Z(T) = \sum_l \rho(l) e^{-E(l)/T} = \sum_l e^{(c-\sigma/T)l}.$$

So $Z(T = \frac{\sigma}{c} \sim \sqrt{\sigma}) \longrightarrow \infty$ and long loops proliferate with $m(T = T_H \equiv \frac{\sigma}{c}) = 0$.

These arguments lead to 2nd order, but ignore interactions between (string-like)Hadrons.

Natural at $N = \infty \Rightarrow$ $\boxed{\text{Is the transition 2nd order at } N = \infty \text{ ?}}$

Clash with reduction :

- $\langle W \rangle$ are independent of L for $L > L_{\text{transition}}$ **recent Neuberger et al.**

  then relabel axes and

- $\langle W \rangle$ are independent of T for $T < T_{\text{transition}}$ **Gocksch and Neri '83**

  If $T_{\text{transition}} = T_d$ then choose $\langle W \rangle = \langle \Box \rangle \sim$ Entropy,

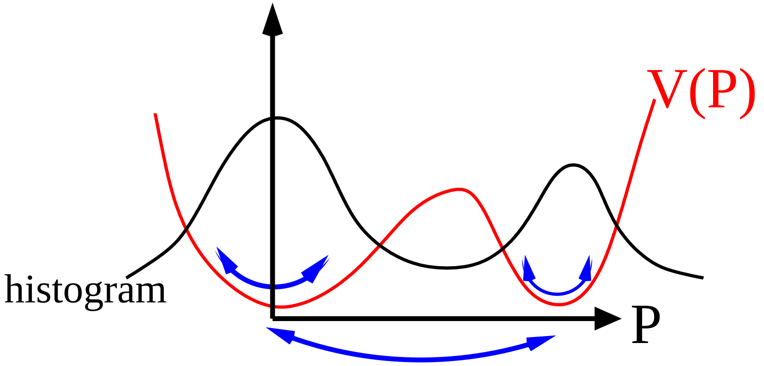
$$\text{Entropy}(\boldsymbol{T_d^-}) = \text{Entropy}(\boldsymbol{0}) \sim \boldsymbol{\mathcal{O}(N^0)}, \text{ but Entropy}(\boldsymbol{T_d^+}) \sim \boldsymbol{\mathcal{O}(N^2)}$$

So what happens on the lattice ?

Numerically : $SU(2)$ **is 2nd** while $SU(3)$ **is weakly 1st**. **Mclarren and Svetitsky '81, Fukigita et al. '89, Brown et al. '88**

Old $SU(4)$: 1st order but near bulk, not continuum $(L_t \leq 4)$. **Svetitsky and Batrouni, Gocksch and Okawa, Gross and Wheater '83**

Confirmed by **Wingate and Ohta '99** $(S_W, < 20^3 6)$ **and Gavai '02** $(S_W^+, L_t = 4-8, L_s \geq 2L_t)$ **,** :



Signs of first order :

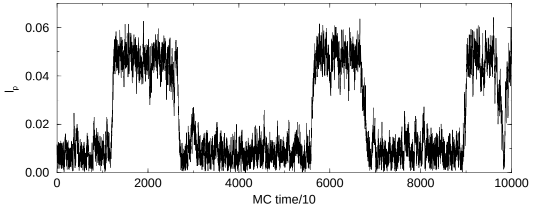
- Double peak structure in histogram of $\langle P \rangle$.
- Tunneling configurations.
- Finite size scaling: $\langle P^2 \rangle - \langle P \rangle^2 \sim V$.

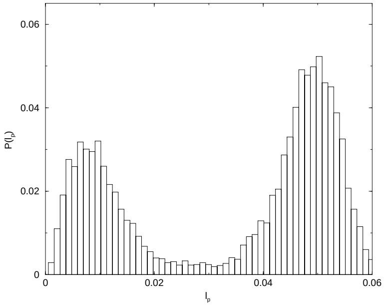
The latent heat :

$$L_h = \Delta\epsilon, \quad \frac{\epsilon}{T^4} = \frac{1}{T^2 V}\frac{\partial \log Z}{\partial T} = \frac{1}{T^2 V}\frac{\partial \log Z}{\partial \beta} \times \frac{\partial \beta}{\partial T} = L_t^4 \langle \Box \rangle \times \frac{\partial \beta}{\partial \log a^{-1}}.$$

Find $L_h = 0.67(2) \times SB$ with perturbative scaling **Higher than $SU(3)$!!!** .

Systematic study of $SU(2 \leq N \leq 8)$ **Lucini, Teper and Wenger '02-'05**





At $\boldsymbol{\beta_d(L_t, L_s)}$ **:**

$$\chi(P) = \langle P^2 \rangle - \langle P \rangle^2 = \text{max.}$$

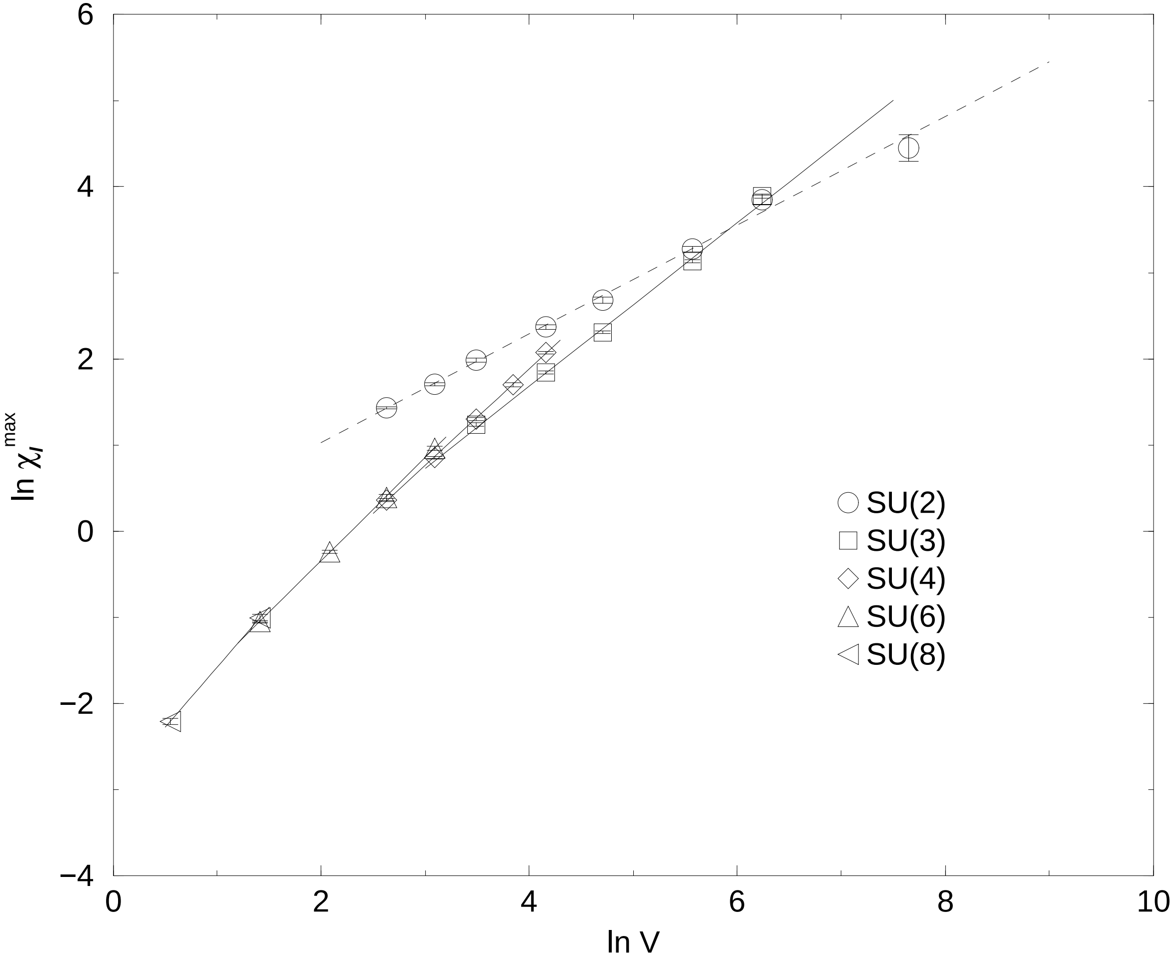
At $\boldsymbol{\beta_d(L_t, L_s)}$ **:** $\chi_{\text{max}} \sim V^{\gamma}$:

$$\gamma = \begin{cases} 1 & 1st \\ < 1 & 2nd \end{cases}$$

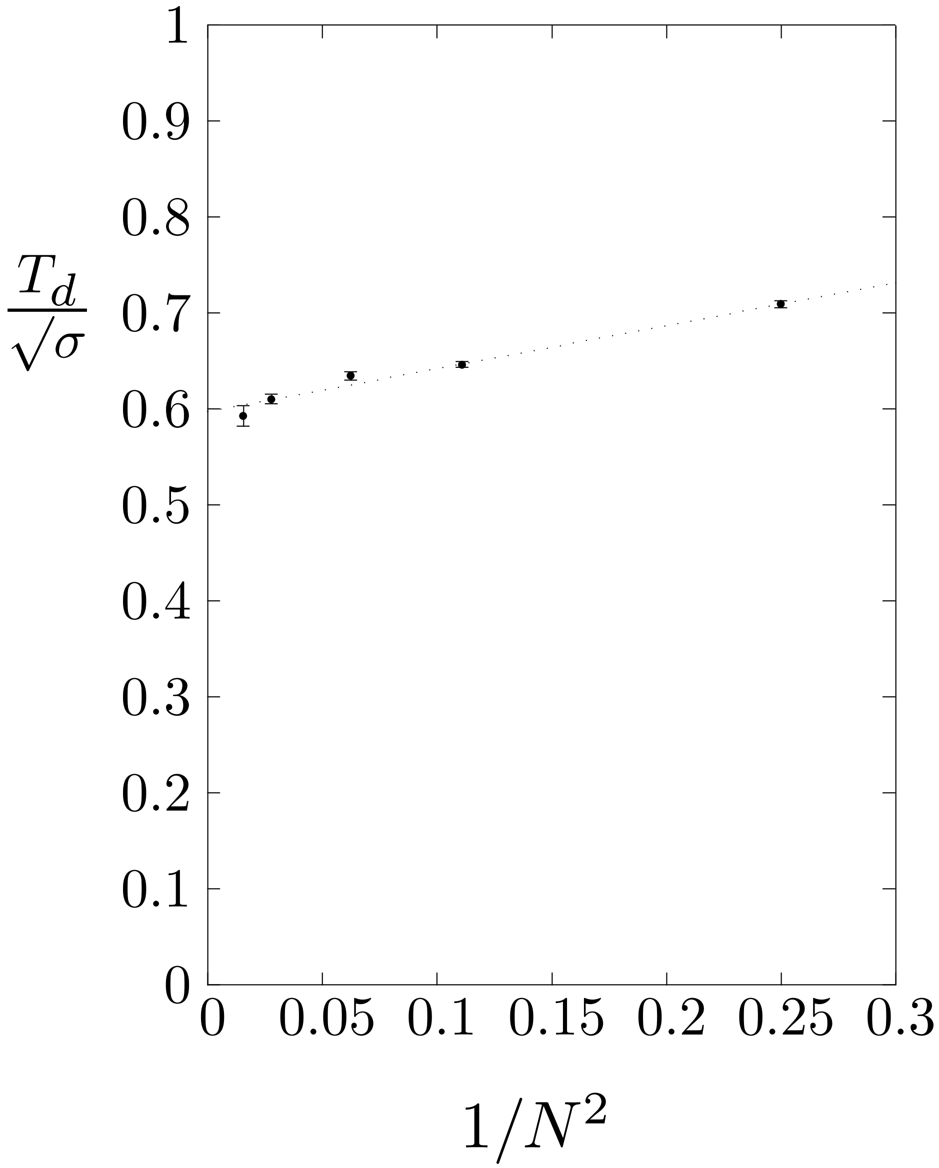
$N = 4 : (12 - 20)^3 5, 16^3 6, 24^3 8$

$N = 6 : (8 - 14)^3 5, 16^3 6, 16^3 8$

$N = 8 : (8 - 10)^3 5, 8^3 6, (10 - 12)^3 8$



And after a $V \to \infty, a \to 0$ get : [use Non-perturbative scaling of $a\sqrt{\sigma}(\beta)$]



- $\boldsymbol{T_d(3) - T_d(\infty) \simeq 8\%}$

- $\boldsymbol{\frac{T_d}{\sqrt{\sigma}}(\infty) = 0.5970(38)}$

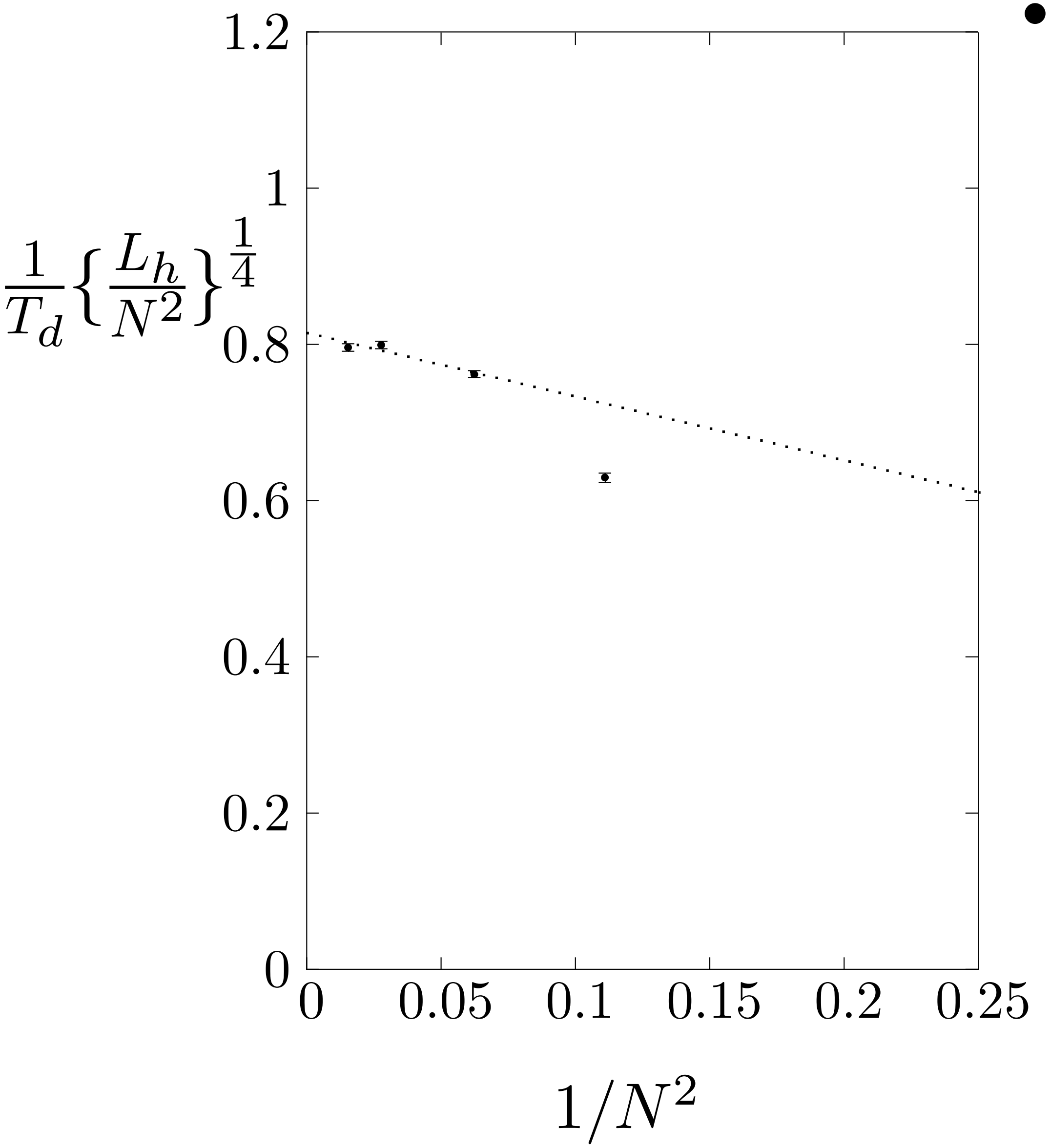
- Neuberger's et al.: $\boldsymbol{\frac{L_d^{-1}}{\sqrt{\sigma}}(\infty) \simeq 0.64}$.

  $5^4 - 10^4$ and $SU(41-23)$

- $V \to \infty$ extrapolation:

$$\boldsymbol{T_d(V) = T_d(\infty) + \frac{\mathcal{O}(1/N^2)}{V}}$$

And after a $V \rightarrow \infty$ but at $a^{-1} = 5T_d$ get : $SU(4,6,8): \ < 32^3 5, 16^3 5, 12^3 5$



- After $a \rightarrow 0$ extrapolation:

$$\frac{1}{T_d}\left(\frac{L_h}{N^2}\right)^{1/4} = 0.766(40)$$

or

$$L_h/S.B. = 0.52(11)$$

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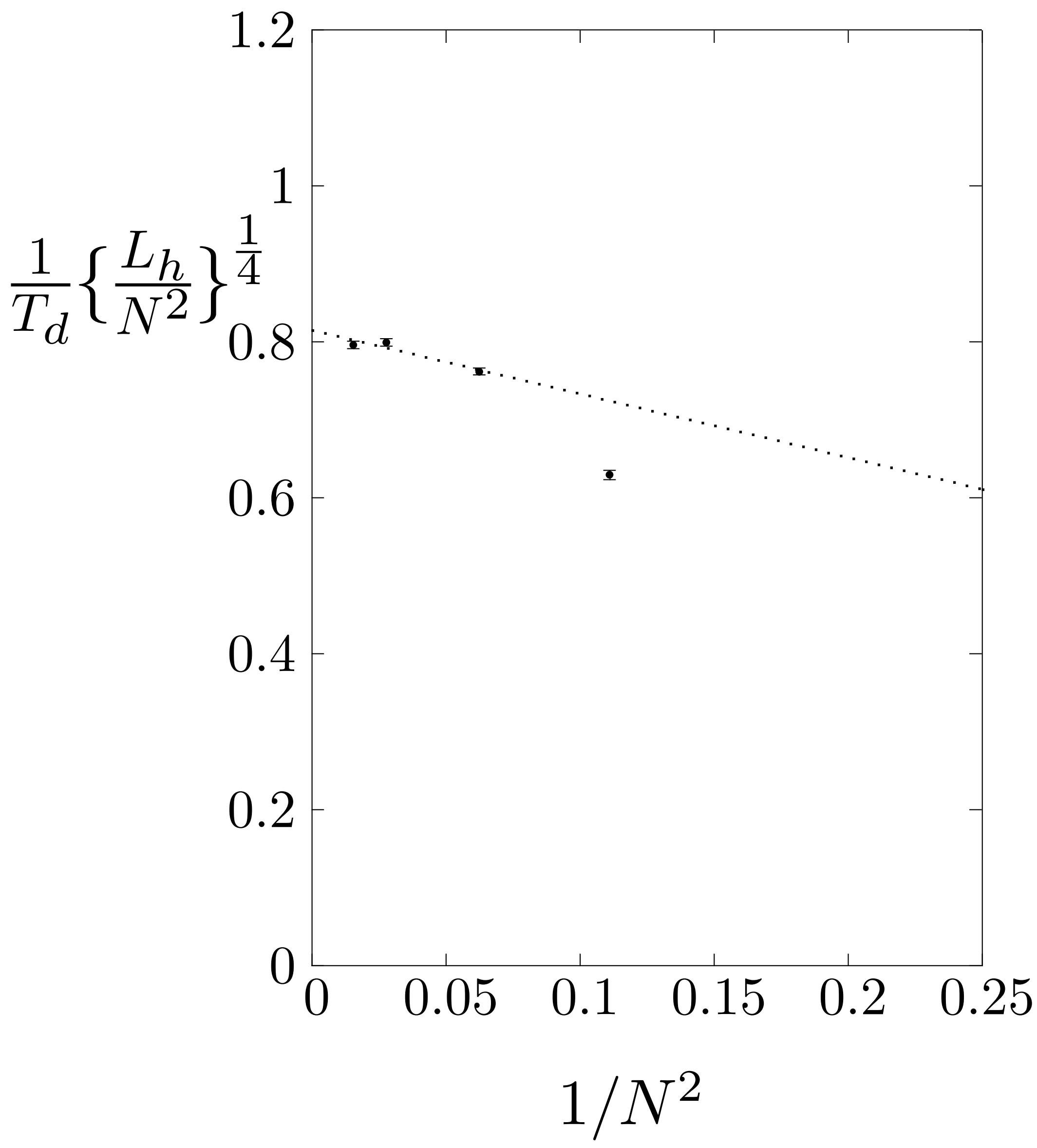

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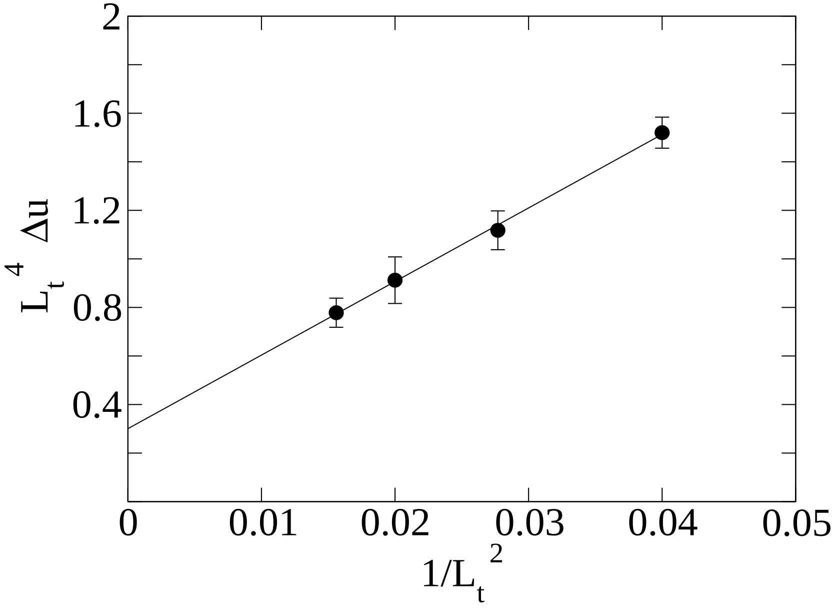
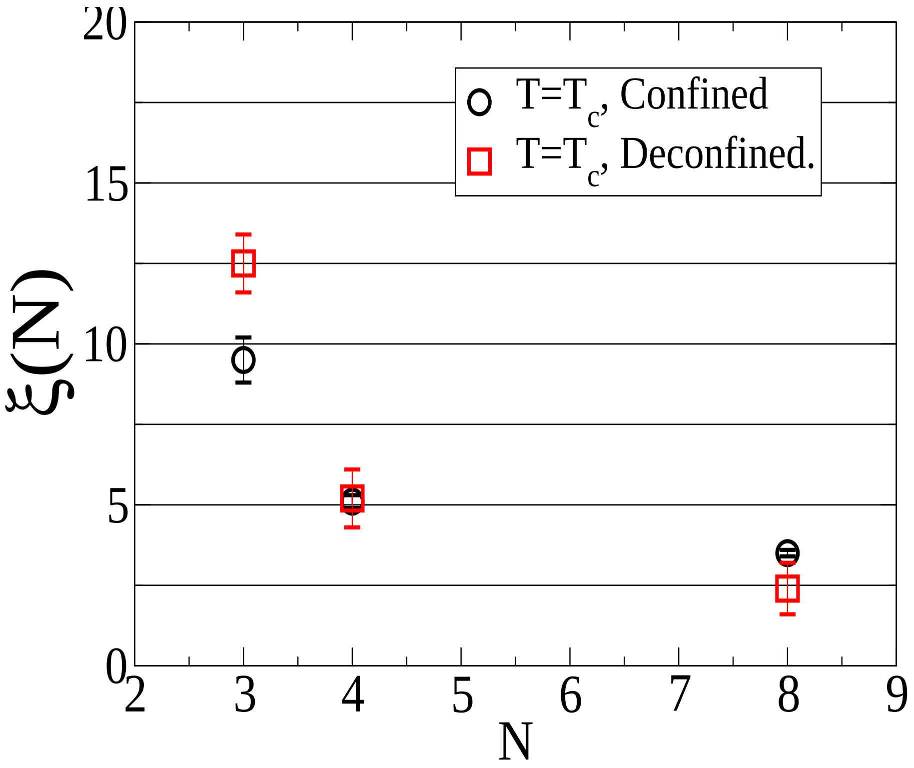
$$\boldsymbol{L_h/S.B. = 0.52(11)}$$

- Kiskis ‘05 : $N = 29, 37, V = 5^4 - 8^4$



$\boldsymbol{L_h/S.B. \simeq 0.26}$ (scaling ?)

From $\langle P_x P_{x+R}\rangle - \langle P_x\rangle\langle P_{x+R}\rangle \sim e^{-m_t R}$ see that at T_d:



$SU(3)$	$SU(4)$	$SU(6)$	$SU(8)$
$64^3 5$	$< 32^3 5$	$16^3 5$	$12^3 5$

Again: expect smaller $1/V$ terms.

Possible because suppressed tunnellings:

$$P(T_d) \propto \exp\left\{-2\sigma_{cd} A/T_d\right\} \sim \exp\left\{-0.03 \times (N T_d L)^2\right\} \sim 10^{-11} \text{ for } SU(12), 12^3 5$$

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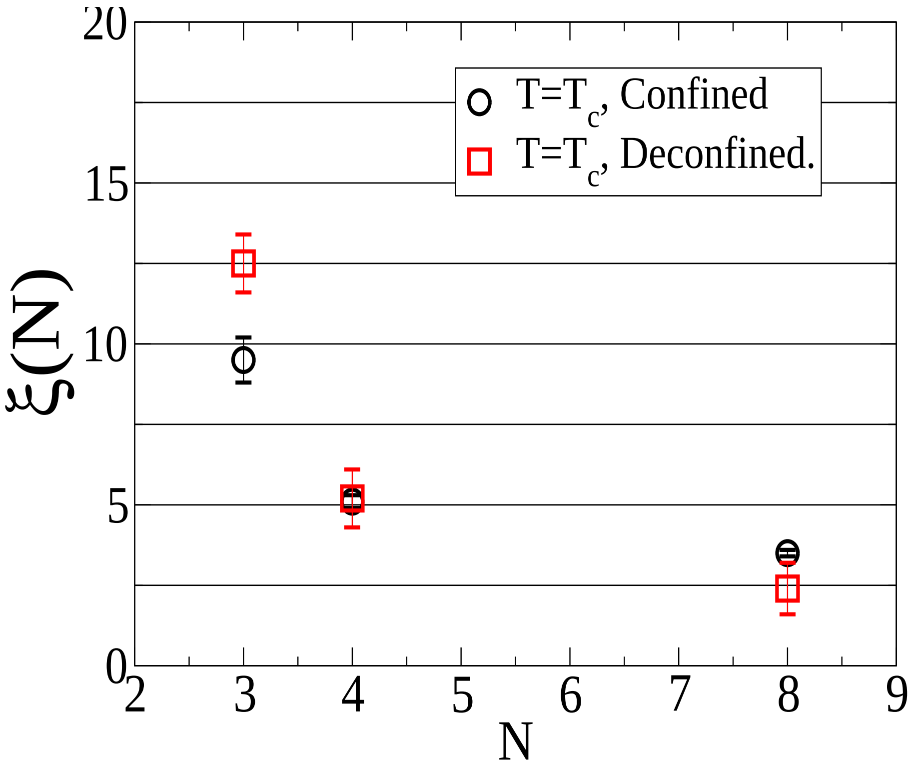

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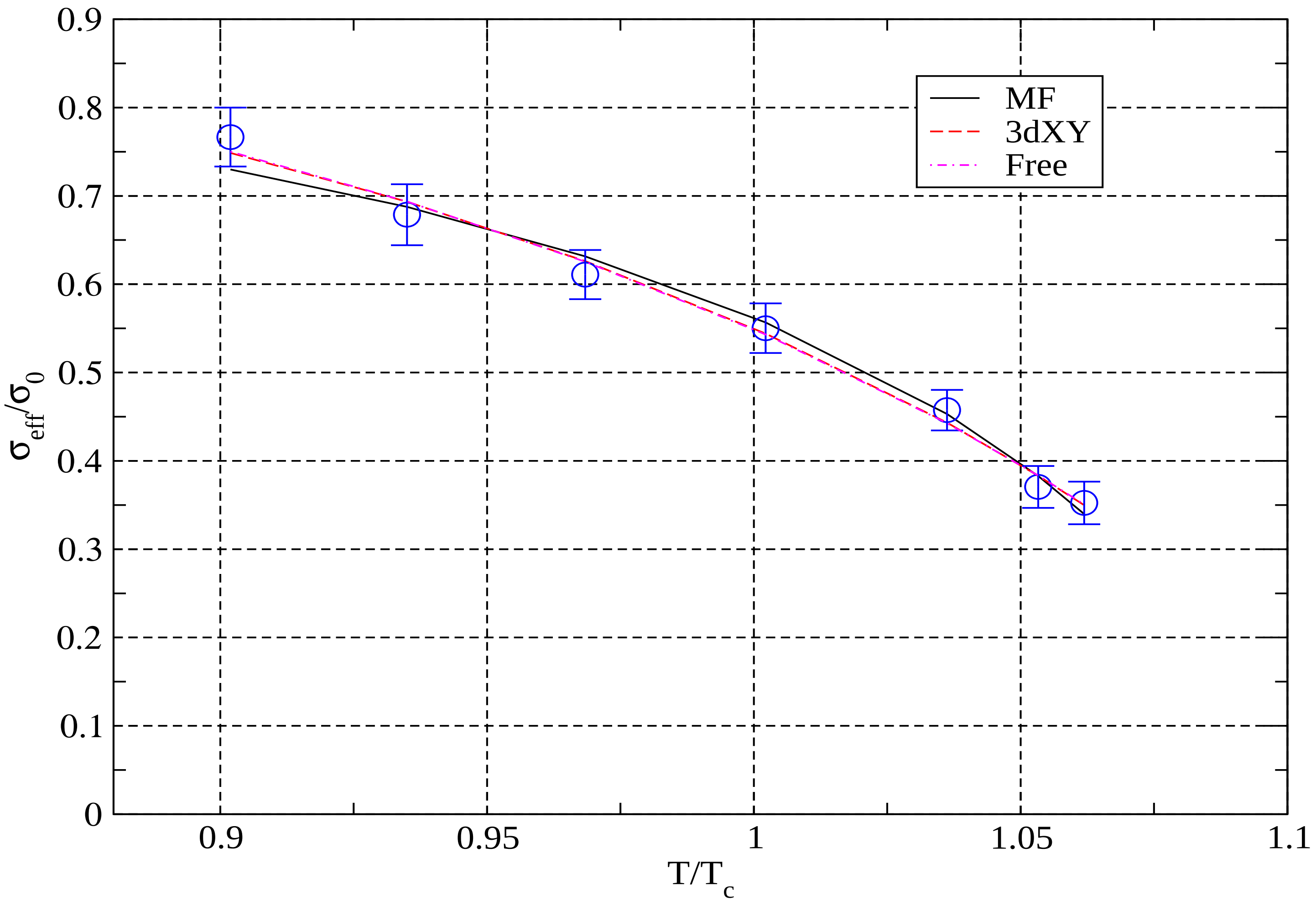
All evidence → deconfinement is 1st order at $N \to \infty$

Where is Hagedorn's T_H ? Deconfinement 'protects' from divergences at $T_H > T_d$.

Numerical MC's : Look at Polyakov-loop mass $\boldsymbol{m_t(T > T_d)}$. **BB and Teper '05**

- $\boldsymbol{m_t(T)}$ decreases with T, extrapolate to $\boldsymbol{m_t(T^*) = 0}$, and identify $\boldsymbol{T^* = T_H}$.
- Do $\boldsymbol{SU(8, 10, 12)}$ on $\boldsymbol{12^3 5}$ (tunneling$\sim 10^{-5} - 10^{-11}$), and find



- $\boldsymbol{\sigma_{\rm eff} = m_t/L_0 \equiv m_t \cdot T}$
- $\boldsymbol{T_H/T_d = 1.092(6)}$, $\boldsymbol{\nu = MF}$
- $\boldsymbol{T_H/T_d = 1.116(9)}$, $\boldsymbol{\nu = 3dXY}$.
- $\boxed{\boldsymbol{T_H/T_d \simeq 1.1}}$

III. $T > T_d$ - Properties of the deconfined phase :

What replaces Hadronic phase after deconfinement ?

Recent years : evidences that for $T_d < T \lesssim 2T_d$, simple QGP is unsuitable.

Lattice : **$10 - 20\%$ deviation from free gas up to $\sim 4T_d$.** **Boyd et al. ‘96**

Wish to comapre other observables to perturbation theory :

- Mass gaps (Debye masses, spatial string tensions).
- Domain wall interface tensions.
- Instantons density.

Large-N can help since

- Perturbation theory becomes simpler (only planar diagrams).
- Physical models should become simpler.

III.A. Bulk thermodynamics on the lattice - BB and Teper ‘05

The integral method : Boyd et al. ‘95,‘96

$$P(T) = \frac{T}{V}\log Z = \frac{1}{a^4(\beta)L_s^3L_t}\int_{\beta_0}^{\beta} d\beta' \underbrace{\frac{\partial \log Z}{\partial \beta'}}_{6L_tL_s^3\langle\Box\rangle_{\beta'}} = \frac{6}{a^4(\beta)}\int_{\beta_0}^{\beta} d\beta' \, \langle\Box\rangle_{\beta'} + [P(T_0)]$$

Regularization : $P(T,\beta) \to P(T,\beta) - P(0,\beta)$ so $\langle\Box\rangle \to \delta\langle\Box\rangle$.

Lattice volumes : 16^35 for $SU(4)$, 8^35 for $SU(8)$, 20^35 for $SU(3)$

Asymmetric lattices $\boldsymbol{T > 0}$: $\to$ $SU(8)$: $\delta\langle\Box\rangle \sim 1\sigma$ from $L_s = 8, 14$.

Is $\langle\Box\rangle_{\boldsymbol{T_d^-}} = \langle\Box\rangle_0$? :

- $SU(3)$: $\mathbf{15}\boldsymbol{\sigma}$ × data from Boyd et al. ‘96
- $SU(4)$: $\mathbf{1.7}\boldsymbol{\sigma}$ much better.
- $SU(8)$: $\mathbf{2.1}\boldsymbol{\sigma}$ much better.

So $\left[\boldsymbol{P/T^4}\right]_{\boldsymbol{T<T_d}} \simeq \mathbf{0}$ for larger N $\to$ small systematics from integration constant.

III.A. Pressure normalized to lattice Stephan Boltzman $(\sim \boldsymbol{N^2})$

$\frac{p/T^4}{\text{free}}$, free$=(N^2-1)\frac{\pi^2}{45}\left[1+\mathcal{O}(1/L_t^2)\right]$ **Boyd et al. '96, Heller and Karsch '84**.

$\downarrow$

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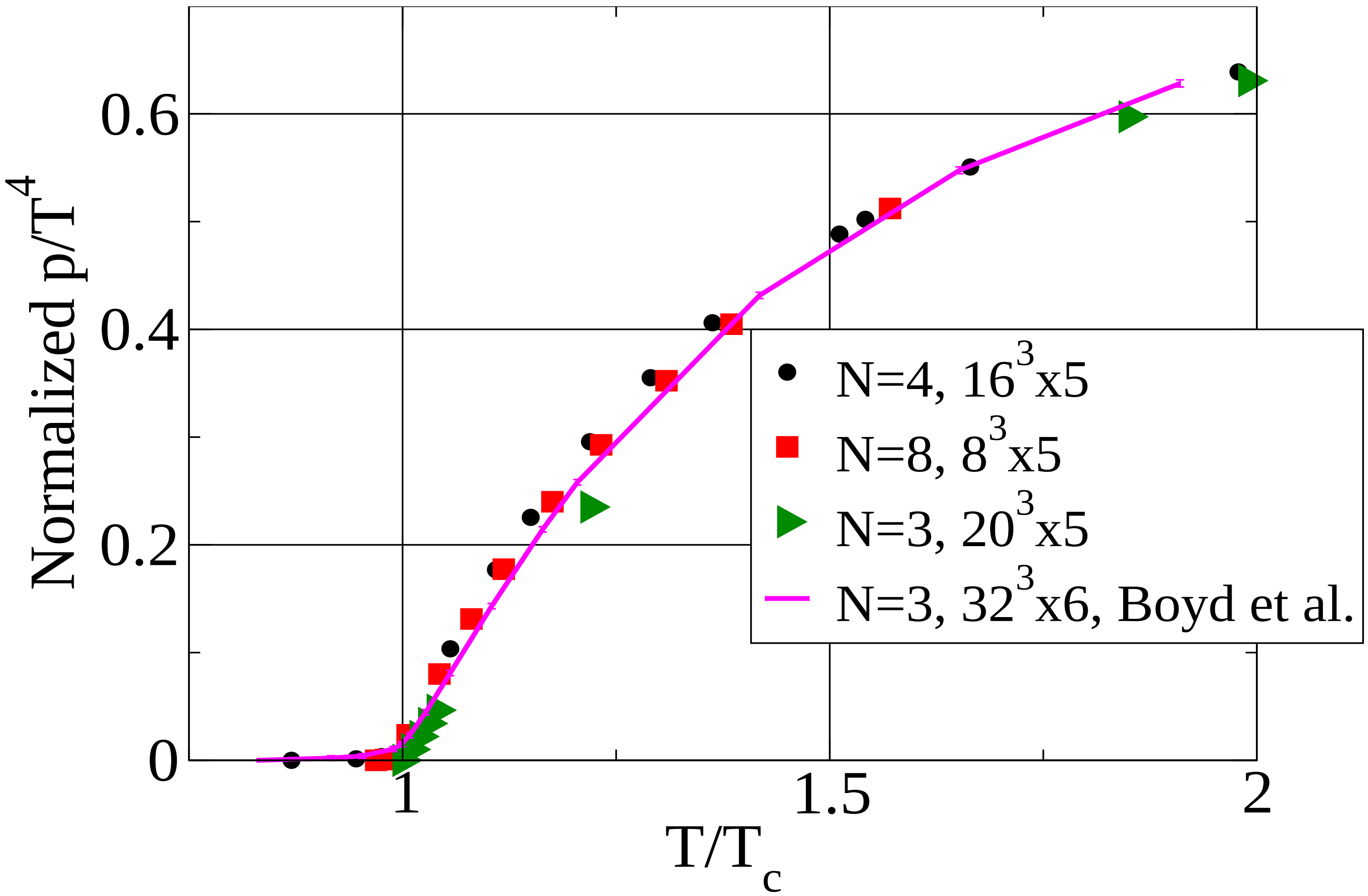
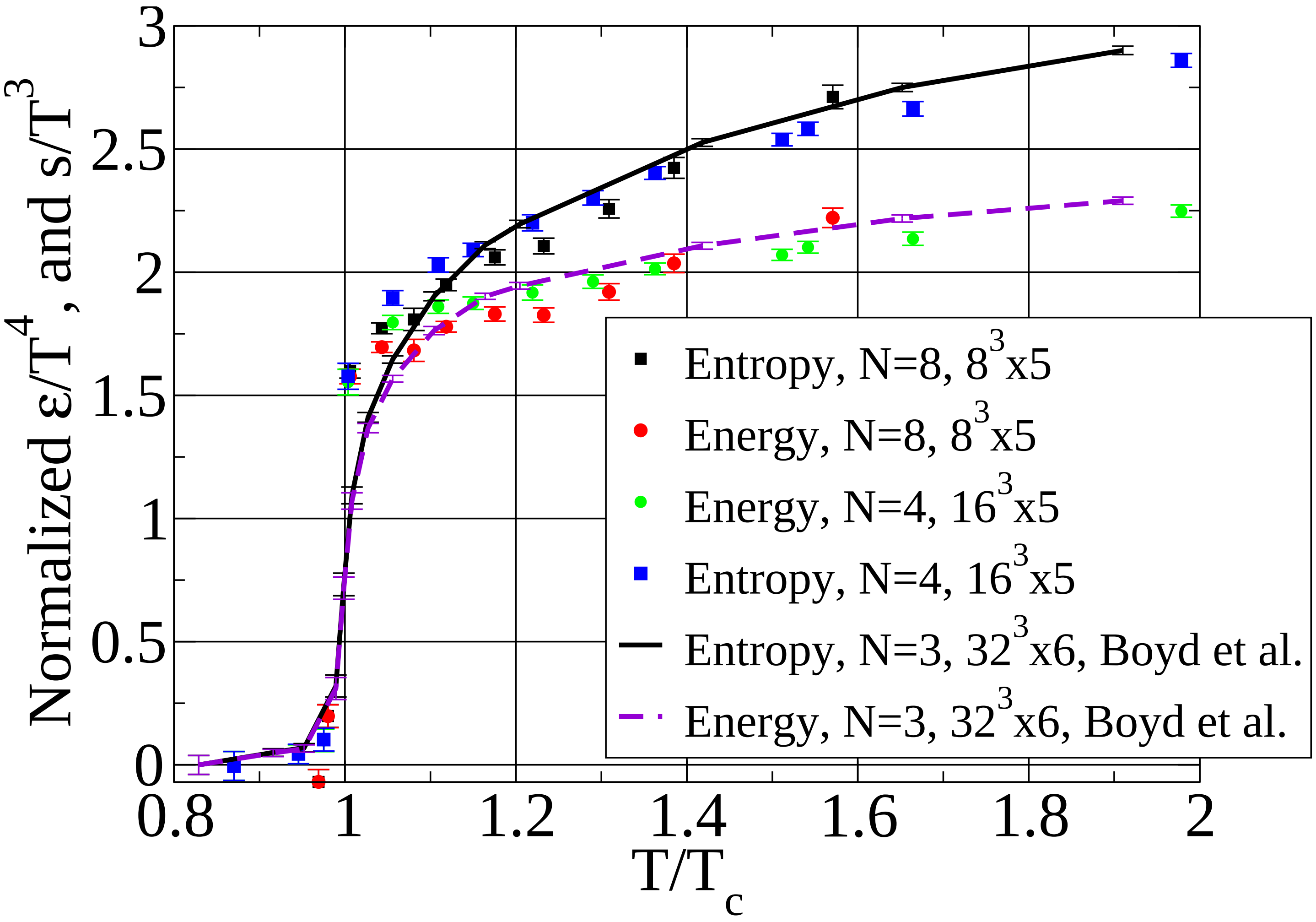


Pressure plots lie almost on top of each other.

III.A. Energy and entropy normalized to lattice Stephan Boltzman ($\sim N^2$)

$\underset{\overset{\downarrow}{\mathbf{4}}}{\frac{s/T^3}{\text{free}}}, \underset{\overset{\downarrow}{\mathbf{3}}}{\frac{\epsilon/T^4}{\text{free}}}$, free$=(N^2-1)\frac{\pi^2}{45}\left[1+\mathcal{O}(1/L_t^2)\right]$ **Boyd et al. '96, Heller and Karsch '84**.

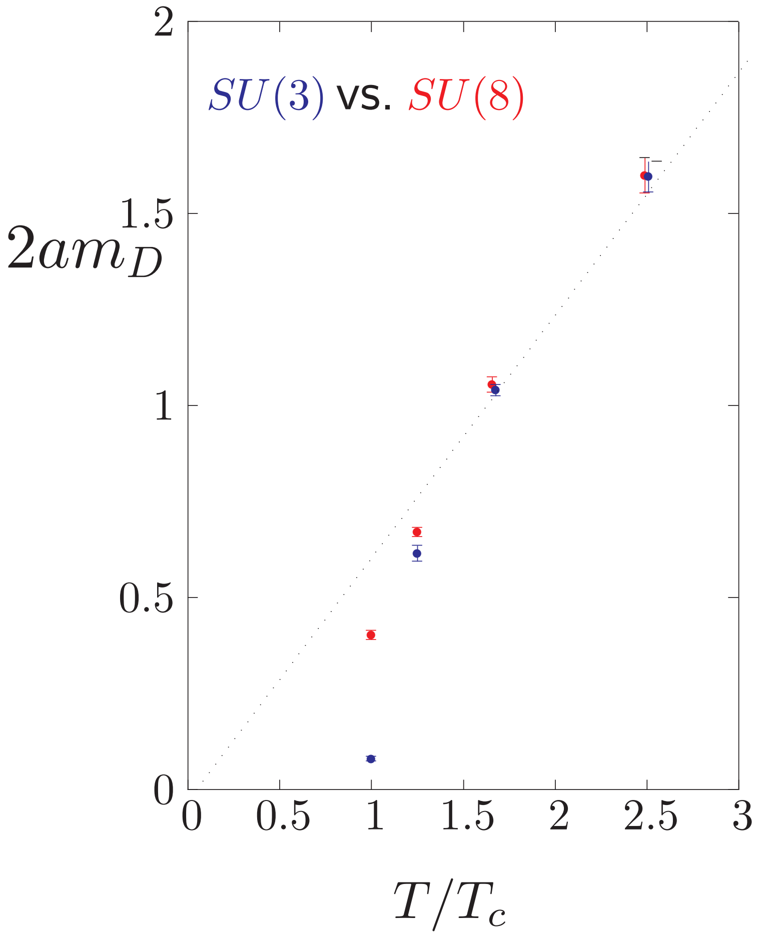
$s/T^3, \epsilon/T^4$ have modest $O(1/N)$ corrections.



III.B. Lattice vs. perturbation theory - Debye screening

Lucini, Teper and Wenger '05

$$\langle P_x P_{x+R}\rangle - \langle P_x\rangle\langle P_{x+R}\rangle \sim \langle A_0^2(x)\, A_0^2(x+R)\rangle \sim e^{-2\, m_D R} \quad ; \qquad m_D = T\sqrt{\frac{g^2(T)N}{3}}$$



- $\boldsymbol{m_D \propto T}$, independent of $\boldsymbol{N}$ for $\boldsymbol{T \geq 1.6 T_d}$.
- $\boldsymbol{\frac{g^2 N(1.6 T_d)}{4\pi} \simeq 0.65}$.

$T = \frac{5T_d}{(2-4)}$ from $8^3(2-4)$.

III.B. Lattice vs. perturbations : spatial masses $\langle P^s_x P^s_{x+R} \rangle \sim e^{-m_s R}$

- $$\begin{aligned} a\sqrt{\sigma}(T=0) &= 0.346(2) \\ a\sqrt{\sigma}(T=T_d^-) &= 0.349(5). \end{aligned}$$

- For $SU(8)$ see that

  $$m_s(T_d^-) - m_s(T_d^+) \simeq 15\%.$$

III.B. Lattice vs. perturbations : spatial masses $\langle P^s_x P^s_{x+R}\rangle \sim e^{-m_s R}$

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  $a\sqrt{\sigma}(T=T_d^-) = \ 0.349(5).$

- For $SU(8)$ see that

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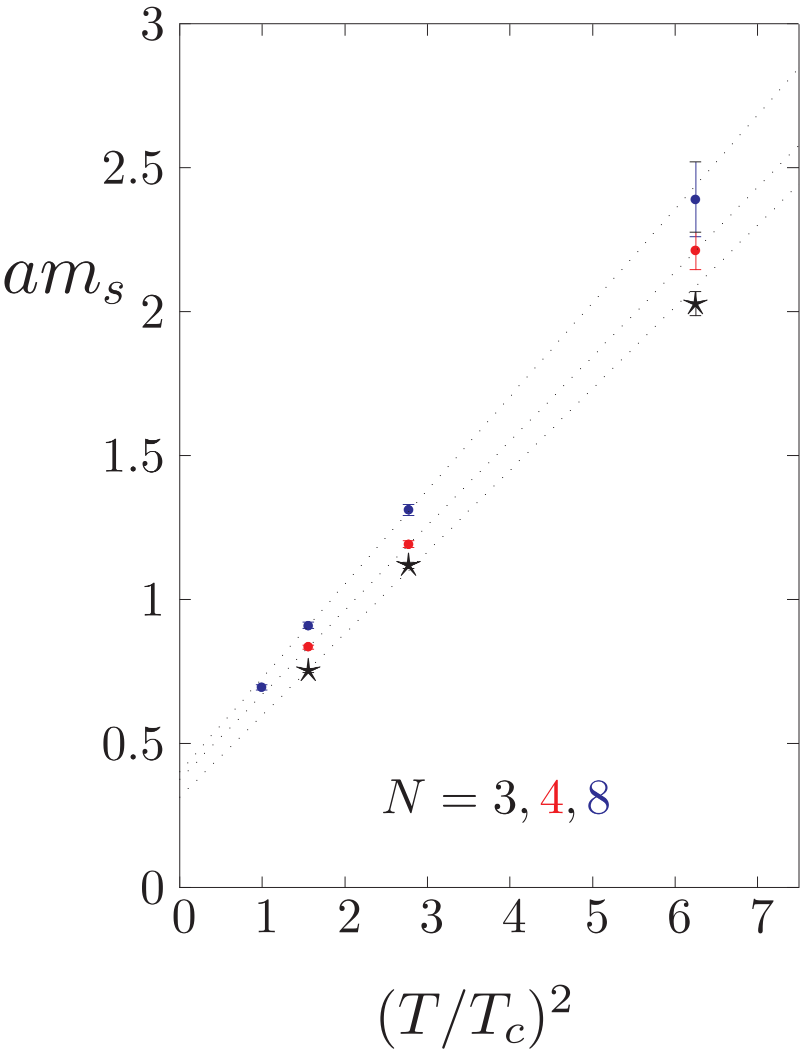
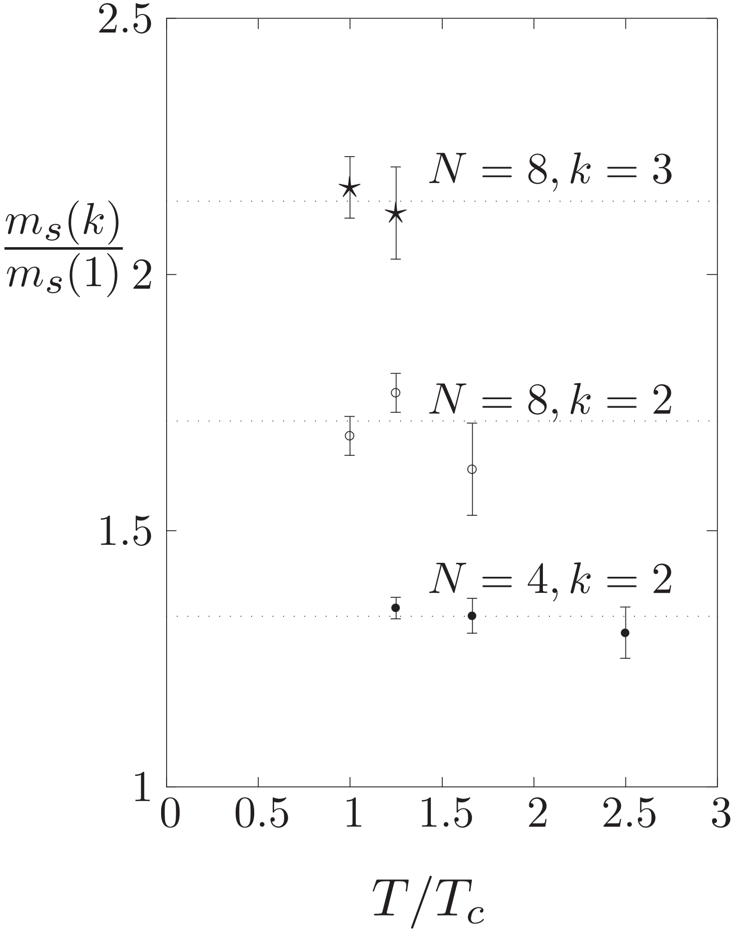
- “Early” large-T behavior.



For $\boldsymbol{T \gg T_d}$ get $\mathbf{2+1}$ with $g_3^2 N \sim g^2 N\, T$ and $\sigma_s(3D) = \left(g_3^2 N\right)^2 \sim \left(g^2 N\, T\right)^2$

III.B. Lattice vs. perturbations : spatial masses $\langle P^s_x P^s_{x+R}\rangle \sim e^{-m_s R}$

Look at $\boldsymbol{k}-$strings : bound states of $\boldsymbol{k}$ fundemental strings, stable for $\boldsymbol{N > 3}$.



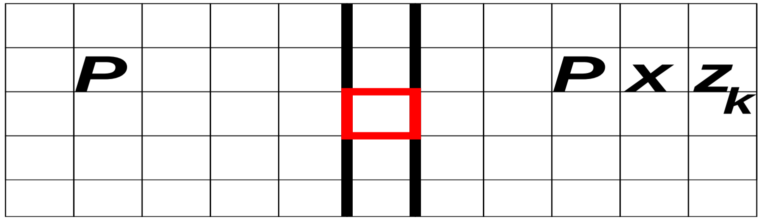
- Casimir scaling strongly preferred over “sine” law.
- Similar to $3D$ rather than $4D$

  → “Early” dimensional reduction ?

- Will be good to compare with perturbation theory a là **Schroder and Laine ‘05**.

III.B. Lattice vs. perturbations : Domain wall tensions

Above T_d : N equivalent g.s.'s separated by domain walls and



$$\frac{\mathcal{Z}_k(A)}{\mathcal{Z}_0(0)} = e^{-F_k(A)/T} = e^{-\Sigma_k \cdot A/T}$$

Perturbation theory : **Korthals Altes and collaborators '92,'01,'04**

$$\Sigma_k = k(N-k)\,\frac{4\pi^2}{3\sqrt{3}}\,\frac{T^3}{\sqrt{g^2(T)N}} + O(g)$$

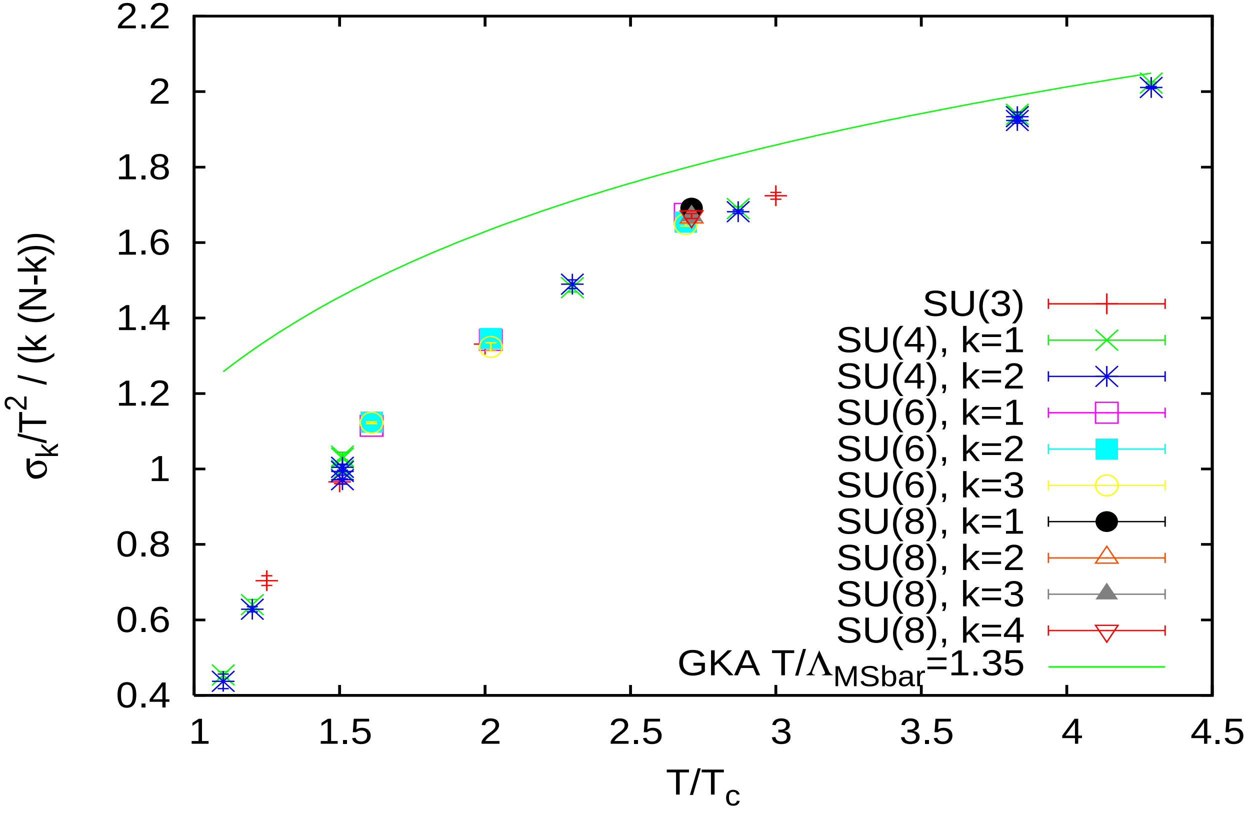
To get Σ_k:

- Enforce a wall by $\beta_\Box \longrightarrow e^{i\frac{2\pi k}{N}}\beta_\Box$.
- Measure $\mathcal{Z}_k(A)/\mathcal{Z}_0(0) = \langle e^{-S(\text{flipped})}\rangle_0 \longrightarrow$ overlap problem.

Two ways out :

- $\partial_\beta\left(\frac{\Sigma_k A}{T}\right) = \partial_\beta\left(\frac{F_k}{T}\right) = \partial_\beta\left(\log Z_0 - \log Z_k\right) = \langle S_k\rangle_k - \langle S_0\rangle_0$ **Bursa and Teper '05**.
- $\frac{\mathcal{Z}_k(A)}{\mathcal{Z}_k(0)} = \frac{Z_k(A)}{Z_k(A-1)} \cdot \frac{Z_k(A-1)}{Z_k(A-2)} \cdots \frac{Z_k(1)}{Z_k(0)} = \prod_{i=1}^{A} e^{-\Sigma_k a^2/T}$ **de Forcrand et al. '04-'05**.

Results : lattice vs. perturbation theory.



de Forcrand, Lucini and Noth '05

$SU(2,3,4,6,8), T/T_d \geq 1.1.\ L_t \geq 5, L_s \geq 3L_t$

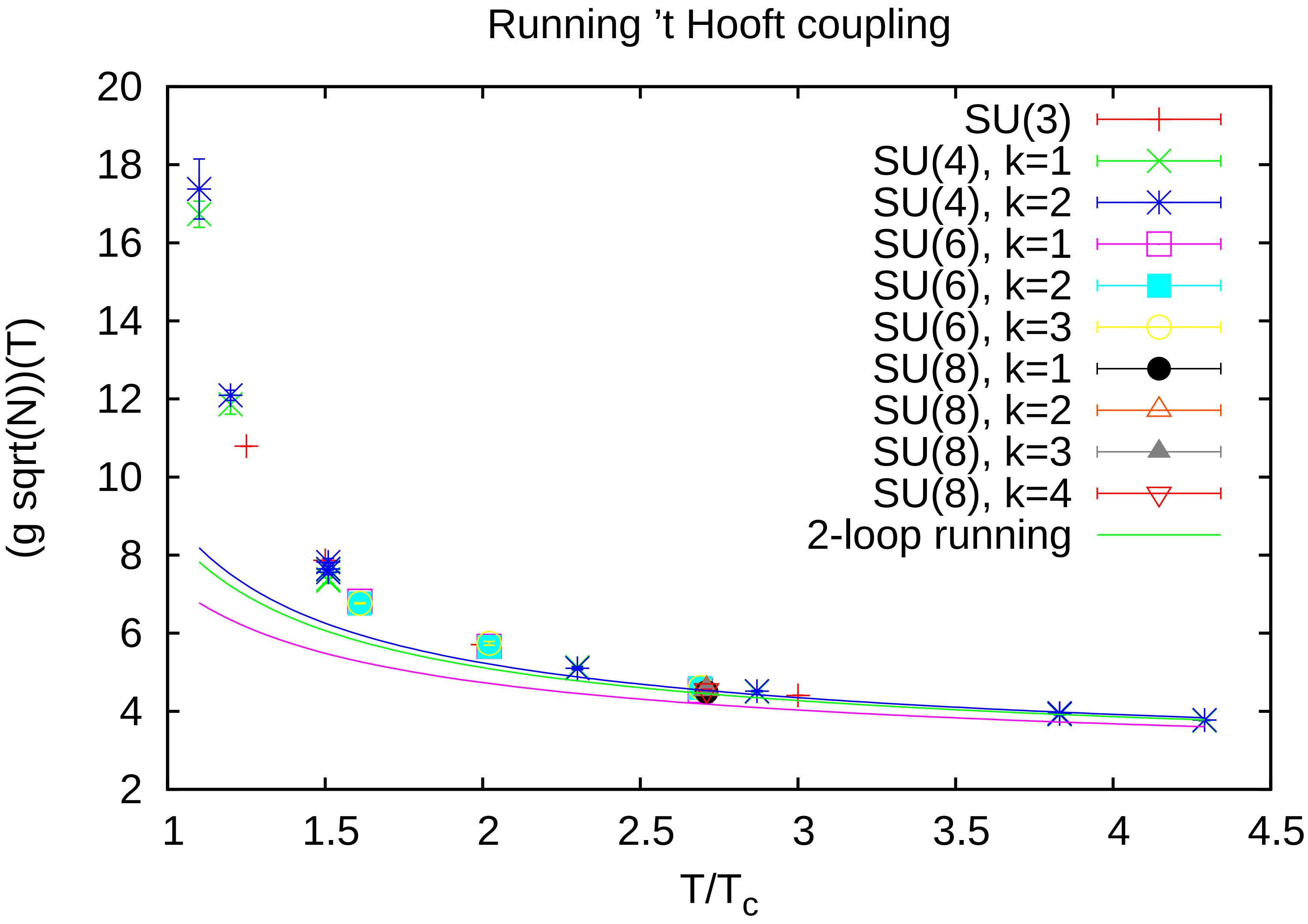
- Casimir scaling for $T \gtrsim T_d$.
- Far from perturbation theory.

Bursa and Teper '05

$SU(2,3,4,6), T/T_d \geq 1.006,\ 20 \times 30 \times (4-5)$

- Same conclusions (but for $\partial_\beta \Sigma_k$)

An interesting exercise : get $g(T)$ from $\Sigma_k(T) = k(N-k)\,\frac{4\pi^2}{3\sqrt{3}}\,\frac{T^3}{\sqrt{g^2(T)N}}$.



Put inside pressure formula : Unfortunately, does not work.

III.C. Topology and intantons

Strong 1st order $\longrightarrow$ calculate $\langle \mathcal{O} \rangle$ in confined and deconfined phase at T_d.

$\longrightarrow$ interesting to calculate $\chi \equiv \langle Q^2 \rangle / V$ with

$$\int d^4x \; Q(x) = \frac{1}{32\pi^2} \epsilon_{\mu\nu\rho\sigma} \int d^4x \; \mathrm{Tr}\, F^{\mu\nu}(x) F^{\rho\sigma}(x) \in Z$$

Instantons above T_d ? $D(\rho) \sim e^{-N \frac{8\pi^2}{g^2 N(\rho)}} \times e^{-N \rho^2 T^2}$ Gross, Pisarski, Yaffe '81

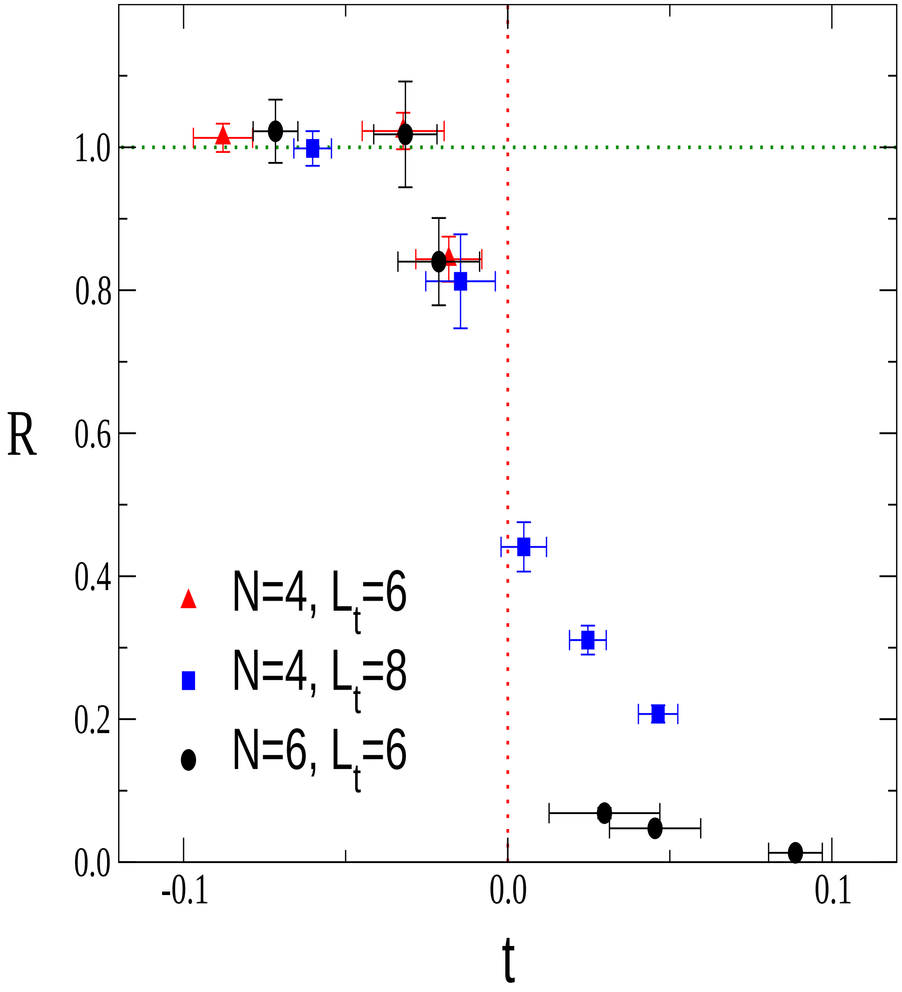
$\longrightarrow$ small and large instantons are suppressed (to $e^{-N} \longrightarrow 0$)

$\longrightarrow$ leaves only $\rho_{\min} < \rho < \rho_{\max}$.

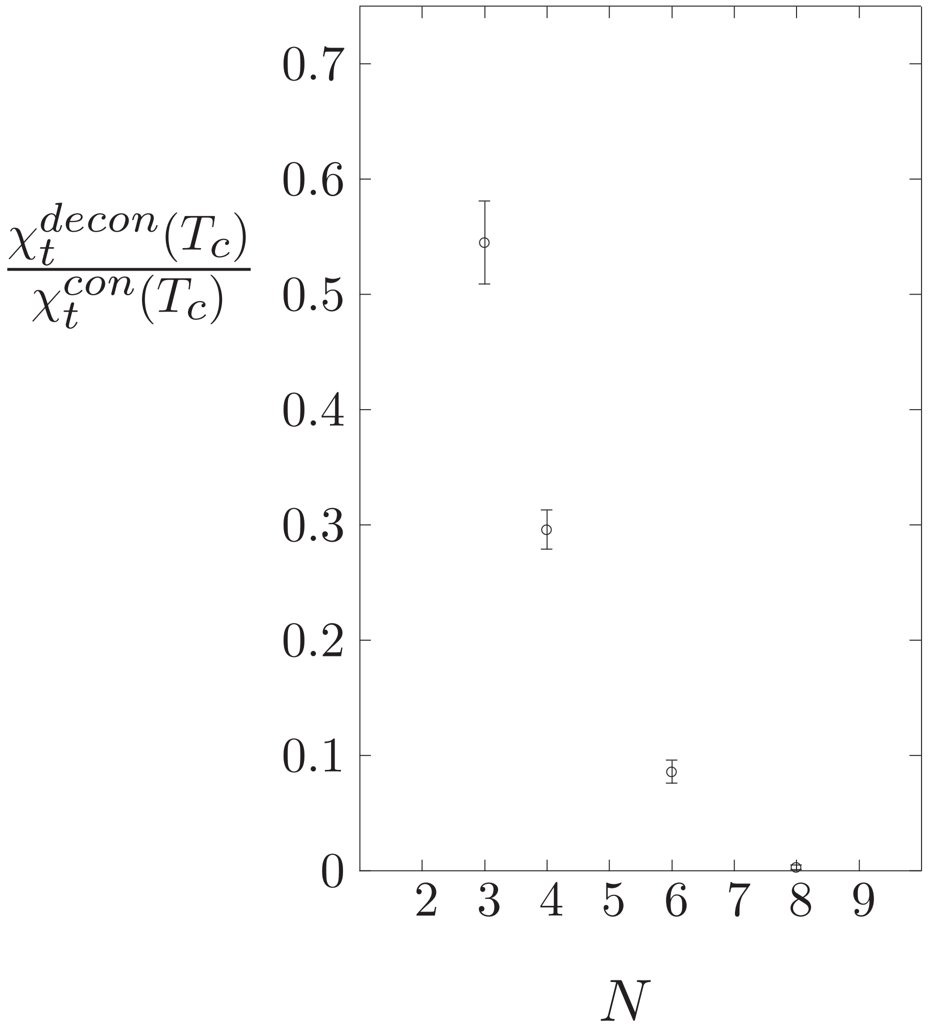
This was checked on the lattice :

- Use "cooling" to smooth gauge configurations.
- Get peak$[Q(x)] = \frac{6}{\pi^2 \rho^4} \longrightarrow$ get $D(\rho)$

Results : **Lucini, Teper, Wegner ‘04,Del Debbio, Panagopoulos, and Vicari ‘04**



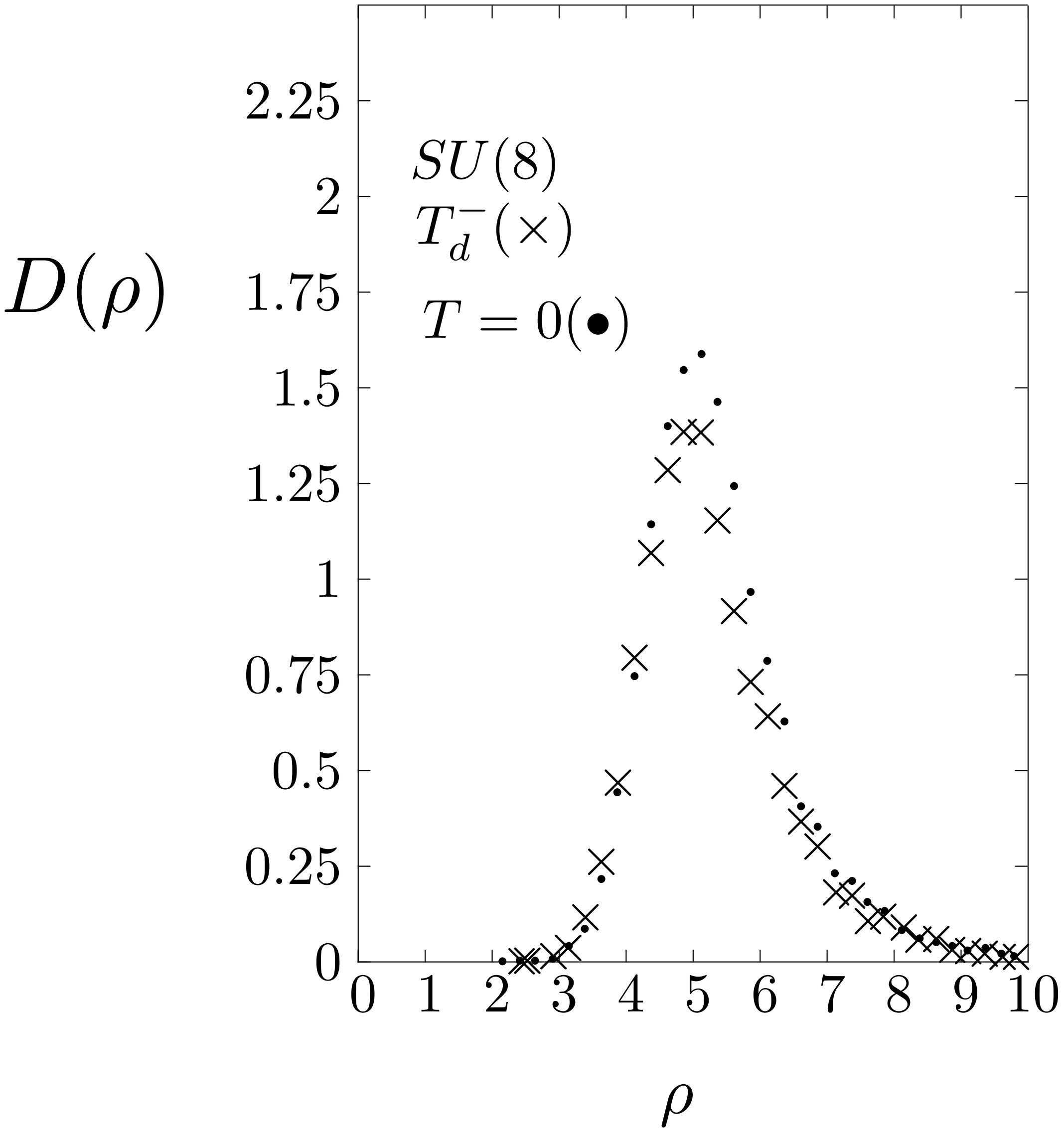
$SU(4,6), L_t = 6, 8, L_s/L_t = 4$

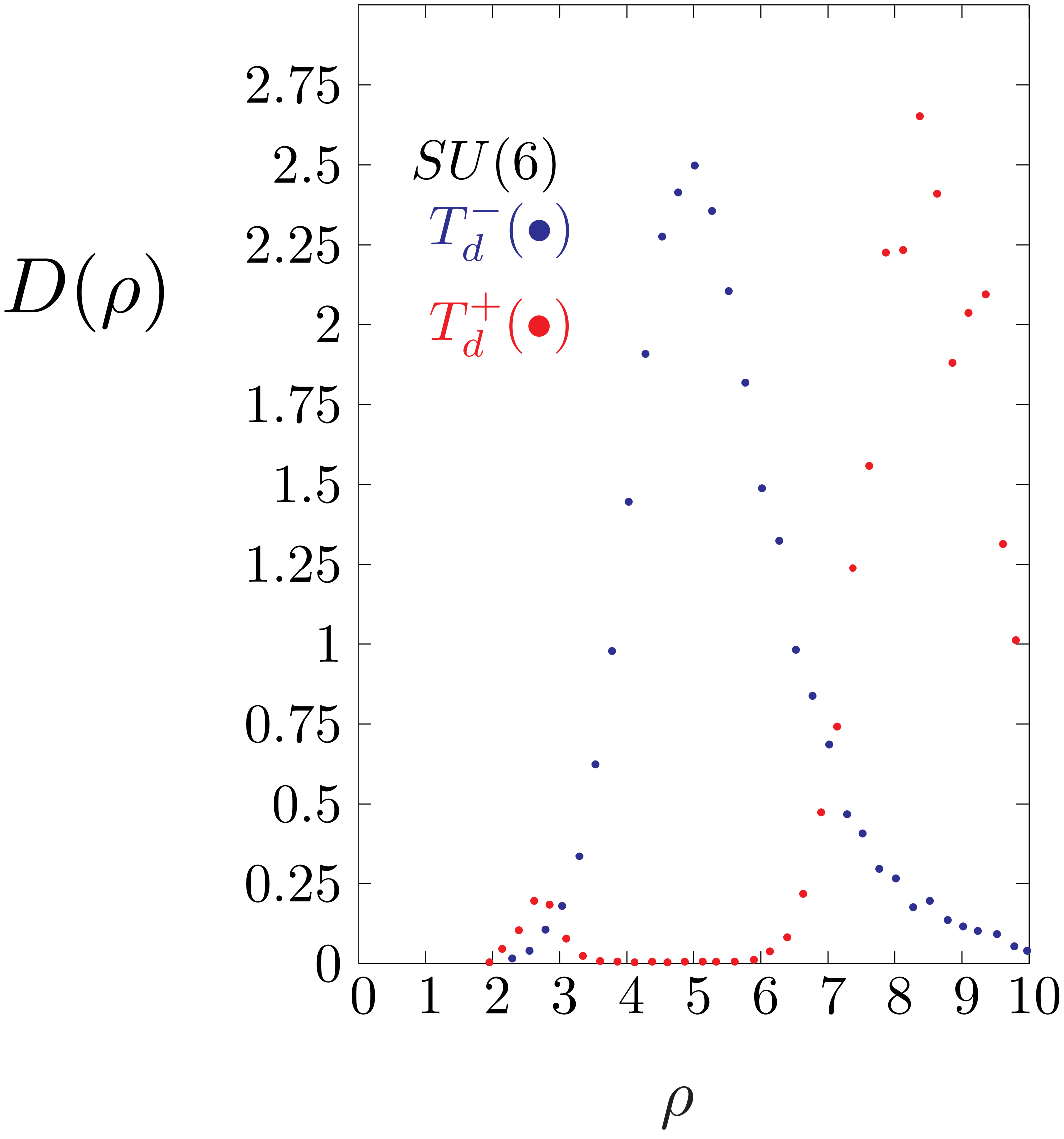


$SU(3)$, $32^3 5$, $SU(4)$, $32^3 5$, $SU(6)$, $16^3 5$, $SU(8)$ $12^3 5$

Find that : $\chi(0) = \chi(T_d^-); \chi(T_d^+) \sim e^{-N} \to 0 \Rightarrow \rho_{\max} < \rho_{\min}$? **Kharzeev et al ‘98**

Check $D(\rho)$:





Similar features in other $SU(N)$.

IV. Summary

What I covered

Large N at $T \simeq T_d$:

- Phase coexistence + finite size scaling.
- $L_h \simeq \frac{1}{2}\, S.B.$.
- $\xi < \infty$ ($\to \infty$ at $T_H \simeq 1.1 T_d$).

$\Longrightarrow$ 1st order transition.

Large N at $T > T_d$:

- "Pressure deficit" survives $N = \infty$.
- Mass gaps - "early" large-T behavior ?
- Interface tensions - Casimir scaling but far from perturbations.
- $\chi^{\infty}_{\text{topology}}(T) = \begin{cases} \chi^{\infty}_{\text{topology}}(0) & T < T_d \\ 0 & T \geq T_d \end{cases}$.

What I did'nt cover but has been done

Wetting phenomenon and σ_{cd} vs. Σ_k.

$SU(N \geq 4)\,@\,3D$ 1st order: **Liddle and Teper '05-'06, Holland '05**

Bulk thermo in $3D$: better approach to perturbation theory **Petersson et al. '06, Liddle '06**

T_d **vs.** T_χ , vacuum alignment at $1/N$. **Narayanan and Neuberger '06**

What can be (and some of it is being) done

(Renormalized) Eigenvalue densities vs. Polyakov loop actions **Dumitru et al. '04, Aharony et al. '04-'06**

Pure gauge on a sphere : **Aharony et al. '04-'06, BB and Wheater in progress**.

Dual Hagedorn transition : area laws for 't Hooft loops below T_d.

Quarkonia at large-N : but first need $T = 0$.

θ **vacua at finite T** a là **Del Debbio, Panagopoulos and Vicari '04,'06**