Color flux-tubes/strings and the Hagedorn temperature : a perspective from the lattice

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Mainly Based on:
BB and M. Teper, hep-th/07xxxxx (preliminary results in hep-lat/0610035)
BB and M. Teper, hep-th/0611286, PLB
A. Athenodorou, BB and M. Teper, work in progress.
BB and M. Teper, PRD (2005) hep-lat/0508021

Outline

Color flux-tubes/strings in $SU(N)$ gauge theories in $2+1$ dimensions

I. **Why $2+1$?**

II. **Flux-tube properties from the lattice** - string masses and string tensions

III. **Results and comparison with theory** - Karabali-Kim-Nair and Nambu-Goto.

IV. **The instability of the flux-tube**

Hagedorn behavior in $SU(N)$ gauge theories in $3+1$ dimensions

V. **Hagedorn's ultimate temperature T_H**

VI. **Seeking for T_H on the lattice**

Summary & future prospects

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Color flux-tubes/strings in $SU(N)$ gauge theories in $2+1$ dimensions

0. **Large-N: reminder of basic (relevant) features**

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II. **Flux-tube properties from the lattice** - string masses and string tensions

III. **Results and comparison with theory** - Karabali-Kim-Nair and Nambu-Goto.

IV. **The instability of the flux-tube**

Hagedorn behavior in $SU(N)$ gauge theories $3+1$ dimensions

V. **Hagedorn's ultimate temperature T_H**

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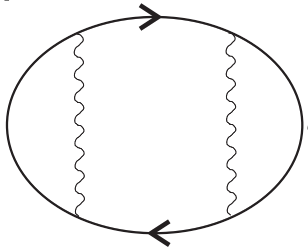
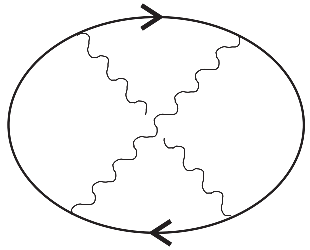
Summary & future prospects

Large N- reminder of basic features:

't Hooft '72 : In $N \to \infty, g^2 N =$fixed limit, use $1/N$ as expansion parameter. Leading diagrams look planar:

E.g. $\langle J(x) J^\dagger(y) \rangle_c$: $\{$  $\sim g^4 N^3 \sim N \}$

$+\{$  $\sim g^4 N \sim 1/N \}$

Witten '79 : Planarity $\to N = \infty$ is a theory of

- Free mesons, free glueballs (no decays, mixings, scatterings).
- Classical, skryrme-like, baryons.

Maldacena '98 : AdS/CFT for large-N gauge theories.

But $SU(\infty)$ still unsolved $\to$ lattice

Allows to check : • Assumptions. • Convergence. • Models' dependence on N.

In the standard approach do $SU(2,3,4,6,8,\ldots)$:

$$\langle O(N) \rangle = \langle O(\infty) \rangle + \frac{a}{N^2} + \frac{b}{N^4} \ldots \Leftrightarrow \text{limit} + \text{corrections.}$$

Teper and co. '98-'06, Del Debbio et. al '02,'06, de Forcrand, Lucini and co. '04,'05

(Also have reduction approach, but we shall not need it here. Eguchi-Kawai and its lattice citations '80, Neuberger & Narayanan '01-'06, Teper and Vahirinos '06)

Many properties have been calculated for large-N :

- Vacuum properties at $T = 0$:
  - Glueball and closed-string/flux-tube spectrum.
  - Topology.
  - Wilson loops.
  - SSB of χS.
- Deconfinement at $T > 0$:
  - T_c.
  - bulk-thermodynamics (pressure etc.)
  - Mass gaps
  - Topology
  - Hagedorn-behavior
  - Restoration of χS.

In the standard approach do $SU(2,3,4,6,8,\ldots)$:

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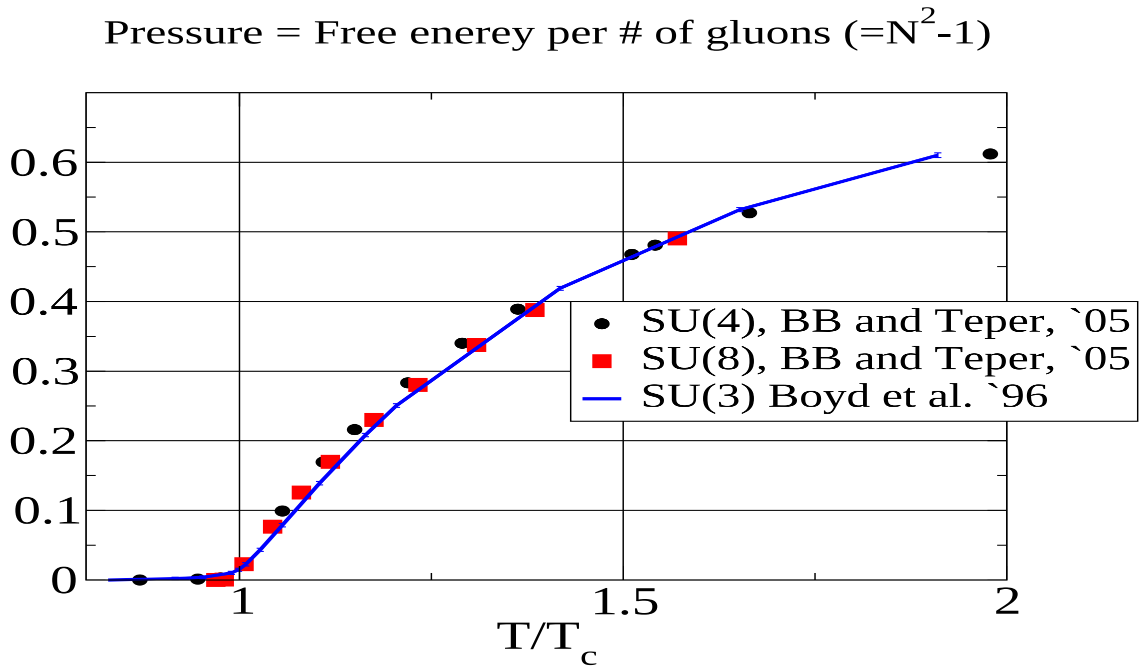
Many properties have been calculated at large-N : usually $SU(3) \simeq SU(\infty)$



Flux-tubes/strings in $SU(N)$ gauge theories in $2+1$ dimensions

Why $2+1$? • Dynamical confinement. • Approx. high-T limit of $3+1$. Kajantie (and many co.)

Karabali-Nair '95 : Continuum YM Hamiltonian + transf. of variables into singlets

$$\frac{\sqrt{\sigma}}{g^2 N} = \sqrt{\frac{1 - 1/N^2}{8\pi}}$$

Within 3% of lattice data !!! Lucini and Teper '02.

The facts that :

- Empirically $N \uparrow$ then discrepancy $\downarrow$
- Karabali-Nair predict no screening - possible only at $N = \infty$.

Exact at $N = \infty$? **If so, Karabali-Nair approach = significant step forward !**

Not seen in '02 : $(KN - \text{lattice}) \overset{N=\infty}{\simeq} +1\%$, but had unestimated systematics.

Our goal : attempt to remove all systematic errors in $\left(\frac{\sqrt{\sigma}}{g^2 N}\right)_{\text{Lattice}}$

String-tensions from the lattice

Define $SU(N)$ gauge theory on a torodial lattice :

- $A_\mu(x) \to U_\mu(x) \in SU(N)$ live on links.
- $S_{YM} \stackrel{\text{discretize}}{\longrightarrow} S(\text{Wilson})$

Four stages in our calculation of string tension :

1. **Calculate the energies of a flux-tube of length L** $\longrightarrow$ **get $E_n(L)$**
2. **From $E_0(L)$ to the string tension $\sigma \equiv \lim_{L\to\infty} \left(\frac{E_0(L)}{L}\right)$** $\longrightarrow$ **get $\sigma(a)$**
3. **To the continuum limit of the $SU(N)$ string tension** $\longrightarrow$ **get $\sigma(0)$**
4. **To the planar limit of $SU(\infty)$ string tension** $\longrightarrow$ **get $\sigma(0)_{N=\infty}$**

1. The spectrum $E_n(L)$ of a flux-tube/string of length L

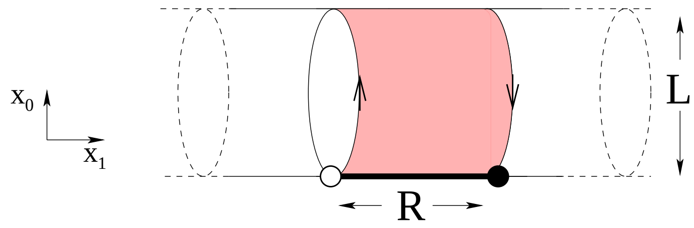
Restrict to closed flux-tubes like Kuti et al., Del Debbio et al.

(In 'open' channel get $V_{q\bar{q}}$: Luscher and Wiesz, Meyer, Cassele et al., Majumadar, Gliozzi et al., Kuti et al.)

Basic idea : Construct lattice operators for closed loops of flux, and measure their mass.

Simplest operator $=$ loop winding around one of the torus directions $=$ 'Polyakov loop'

$$P(x) = \mathrm{tr}\ P \exp\left(i \int_0^L dt\, A_0(x)\right) = \mathrm{tr} \prod_{x \in C} U_x$$

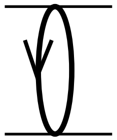


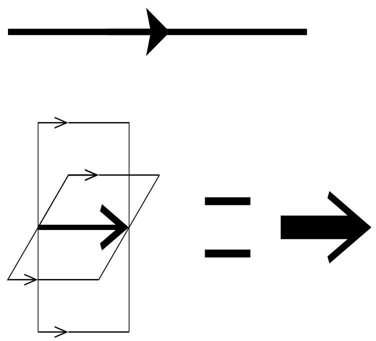
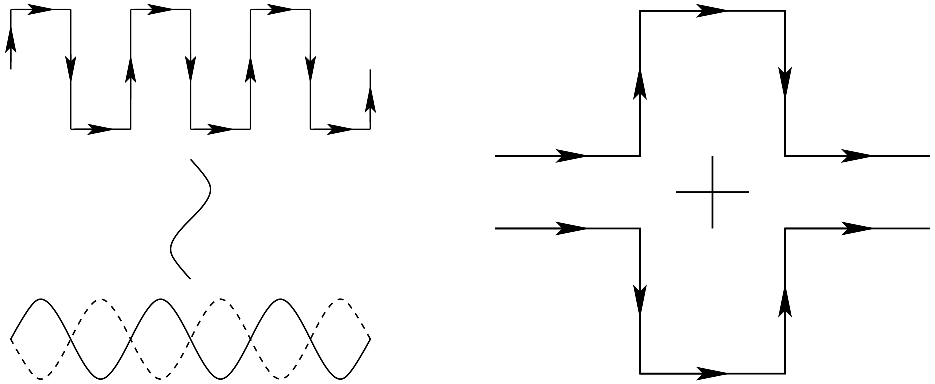
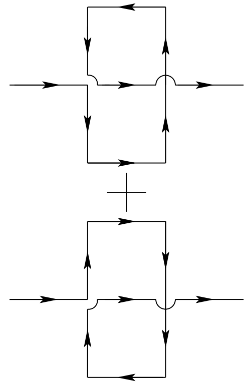
$L \leftrightarrow 1/T$ and so can calculate for $T < T_d$ or $L > L_d$.

Take loops of $\boldsymbol{p_\perp = 0}$ and

$$\langle P(x_1) P^\dagger(x_1 + R) \rangle = \sum_n w_n\, e^{-E_n^{\mathrm{loop}}(L)\, R}$$

Ideal: build ∞ operators, diagonalize ∞ corrlation matrix $M_{IJ} = \langle P_I(x) P_J^\dagger(x + R) \rangle$.

Operators for correlators : $P(x) =$  $= \mathrm{tr} \prod_{x \in C} U_x +$ "Smearing" $+ (5 - 100)$ different operators

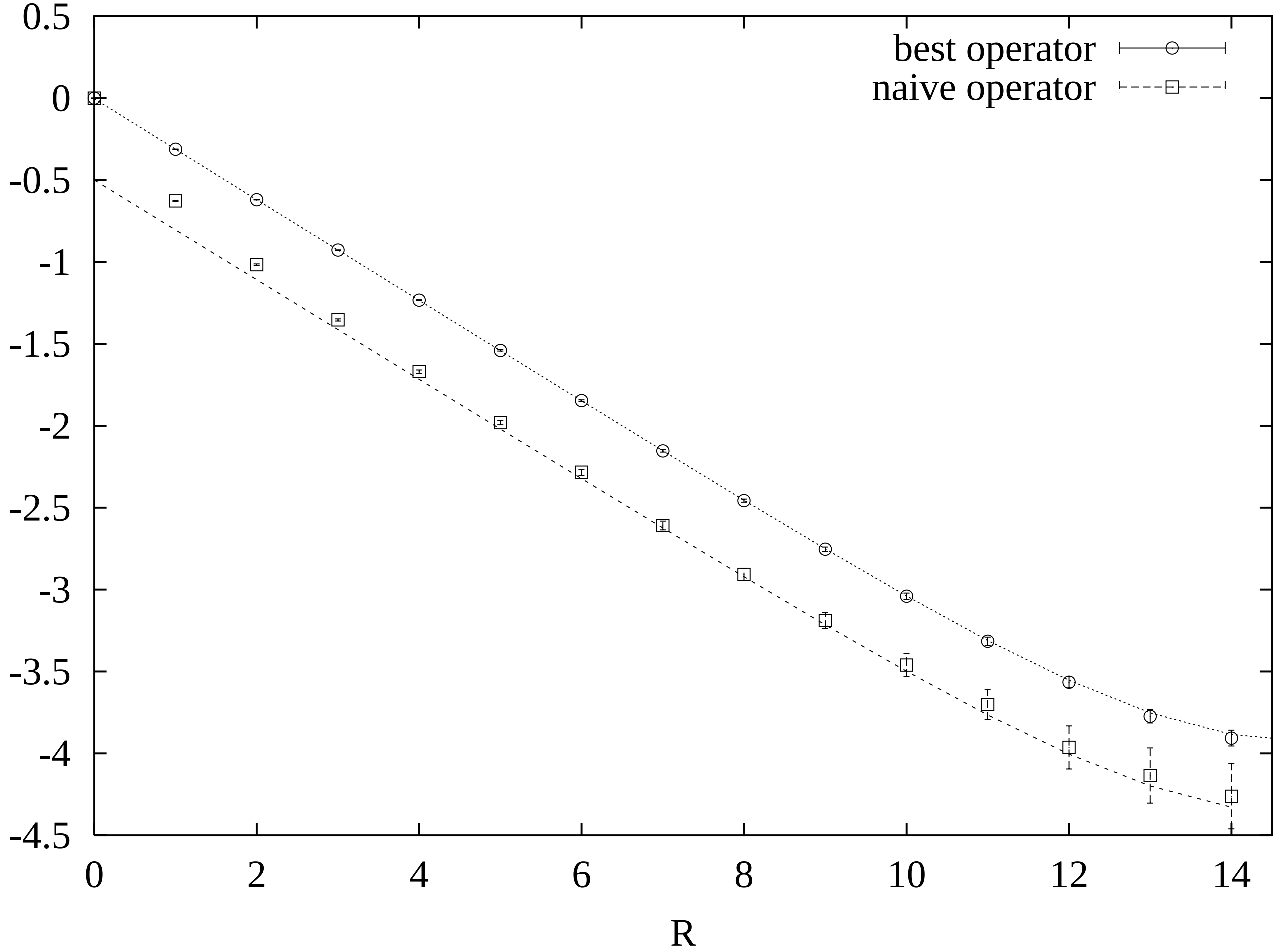
Ground state	Excited states	String-Glueball bound state (?)
		

Eigenvectors approximates the state $|n\rangle$ and so

$$\langle \mathcal{O}_n(x) \mathcal{O}_n^\dagger(x+R) \rangle = (\sim 1) \cdot e^{-E_n R} + (\sim 0) \cdot \sum_{m \neq n} e^{-E_m R}$$

Example : $SU(6)$, $a^{-1} \simeq 2.6$ **'GeV'**, $a\sqrt{\sigma} \simeq 0.17$, **ground state, string length** $L = 12a$

$$\log\left(\langle\mathcal{O}_{\rm gs}(x)\mathcal{O}^\dagger_{\rm gs}(x+R)\rangle\right) = -E_{\rm gs}\,R + \text{(excited state contemination)}$$



Indeed typically find $\langle\mathcal{O}_{\rm gs}(0)\mathcal{O}^\dagger_{\rm gs}(R)\rangle \simeq 0.995 \cdot e^{-E_0 R} + 0.005 \cdot e^{-E_1 R}$

2. From $E_0(L)$ to the string tension $\sigma \equiv \lim_{L\to\infty} \left(\frac{E_0(L)}{L}\right)$

If the low energy flux-tube is described by a string then is:

Luscher, Symanzik and Weisz '80, Luscher '80, Forcrand et al. '85

$$E_0(L) = \sigma L - \frac{\pi(D-2)}{6L} + \left(\frac{1}{L^2}\right) \times \text{?}$$

Second term - zero point energy of $(D-2)$ tranverse oscillations $\rightarrow$ **universal.**

But if ? $\sim 1\%$ then need to control it !

$\longrightarrow$ **Perform a precise measurement of $E_n(L)$ and fit for the beyond Luescher terms**

.

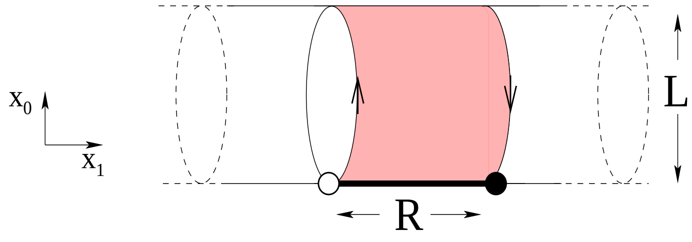
Higher order terms **not universal** - depend on string model.

$\longrightarrow$ obtain valuable information on the effective string theory

Beyond the Luscher term :

$$E_0 = \sigma L - \frac{\pi(D-2)}{6L} + \mathcal{O}\left(\frac{1}{L^2}\right)$$

Nambu-Goto a là Arvis '83 , Action = Area(worldsheet)



$$E_n(L) = \sqrt{(\sigma L)^2 + 8\pi\sigma\left(n - \frac{D-2}{24}\right)}$$

Luscher and Wiesz '02,'04 : Modify NG a là χPT + open-closed loop duality.

$$E_n(L) = \sqrt{(\sigma L)^2 + 8\pi\sigma\left(n - \frac{1}{24}\right) + O\left(\frac{1}{L^3}\right)} \quad \text{at} \quad D = 3$$

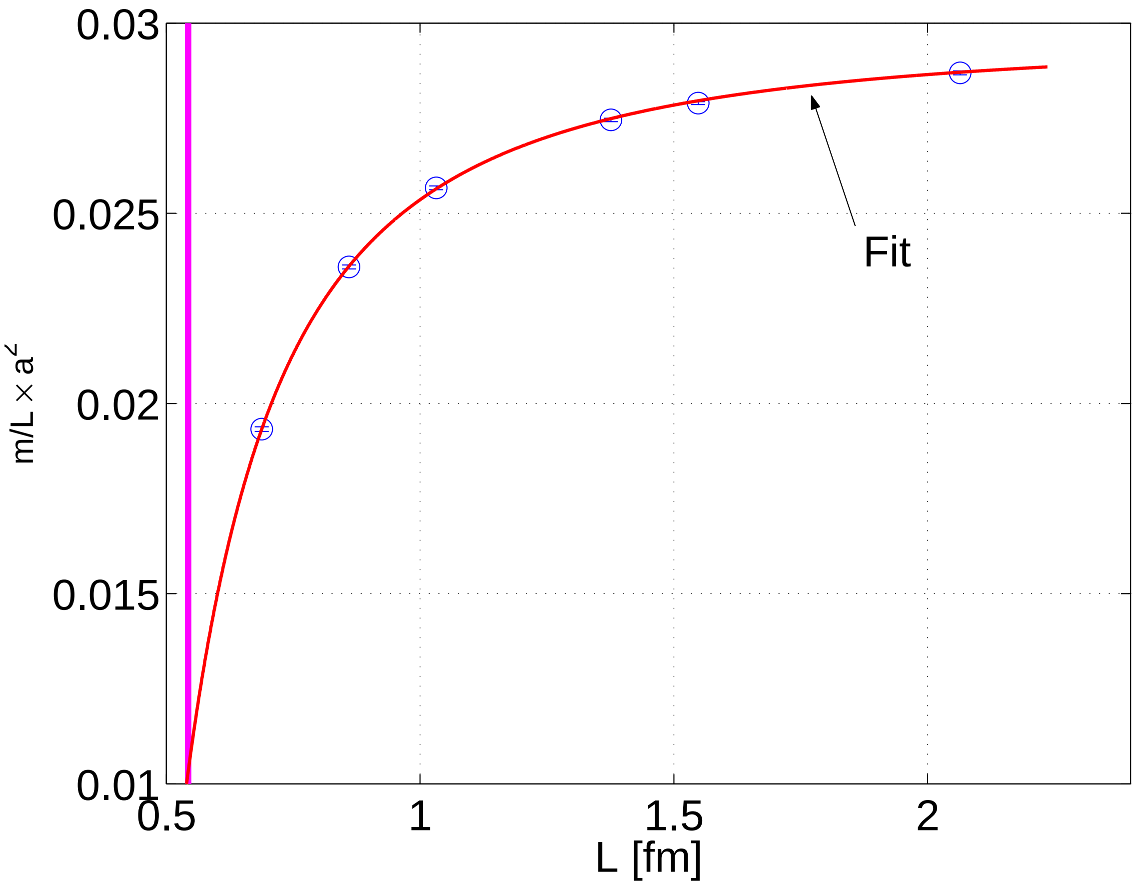
Follow Polchinski & Strominger '91 : Modify NG, remove Weyl anomaly Drummond '04,'06, Dass. and Matlock '06

$$E_n(L) = \sqrt{(\sigma L)^2 + 8\pi\sigma\left(n - \frac{D-2}{24}\right) + O\left(\frac{1}{L^3}\right)} \quad \forall D$$

Results - beyond the Luescher term

$$E_0 = \sigma L - \frac{\pi(D-2)}{6L} + \mathcal{O}\left(\frac{1}{L^2}\right)$$

Do $SU(N)$ with $N = 3, 4, 6, 8$ and $a^{-1} \simeq 1.6, 2.6$ 'GeV' $a\sqrt{\sigma} \simeq 0.25, 0.15)$



- Ground state.
- $SU(6)$, $a^{-1} \simeq 2.6$GeV, $a\sqrt{\sigma} \simeq 0.17$.
- $E_0 = \sqrt{\underbrace{(\sigma L)^2 - \frac{\pi\sigma}{3}}_{\text{NG}} - \frac{C}{\sqrt{\sigma} L^3}}$

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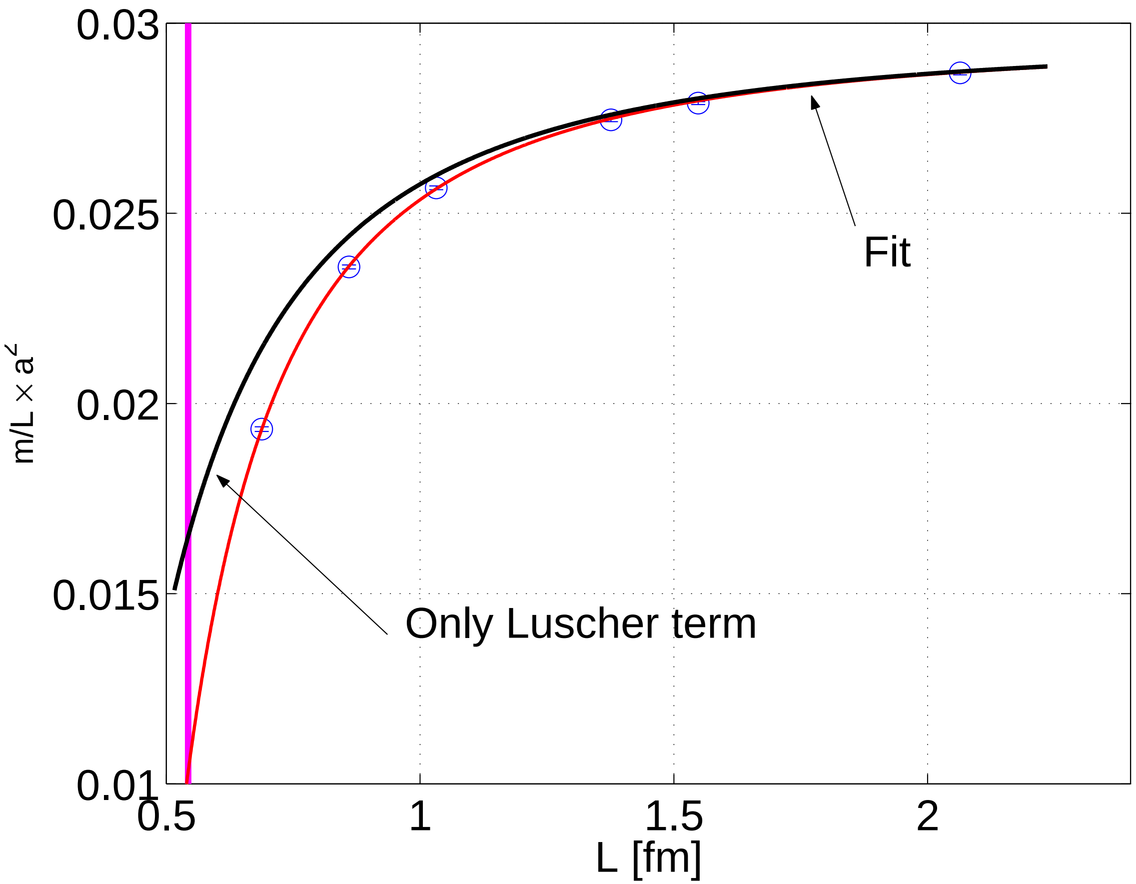

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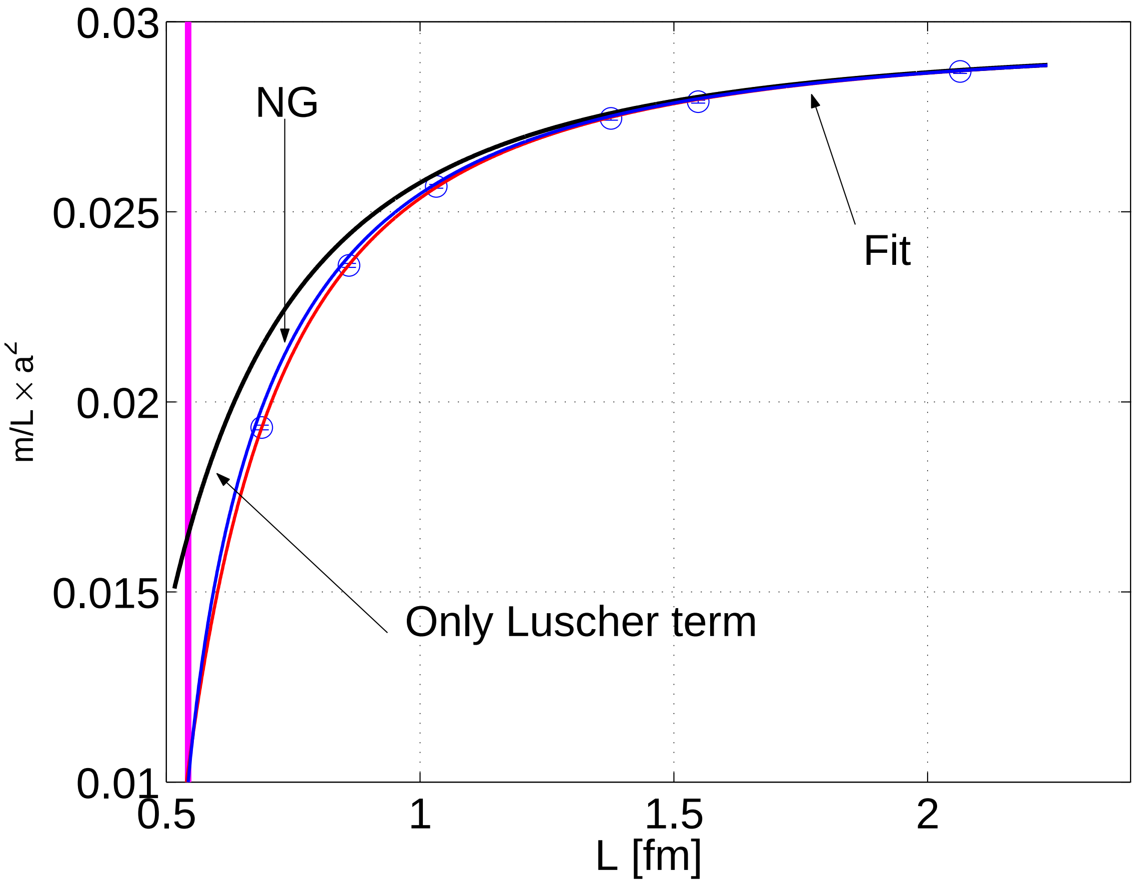

- Ground state.
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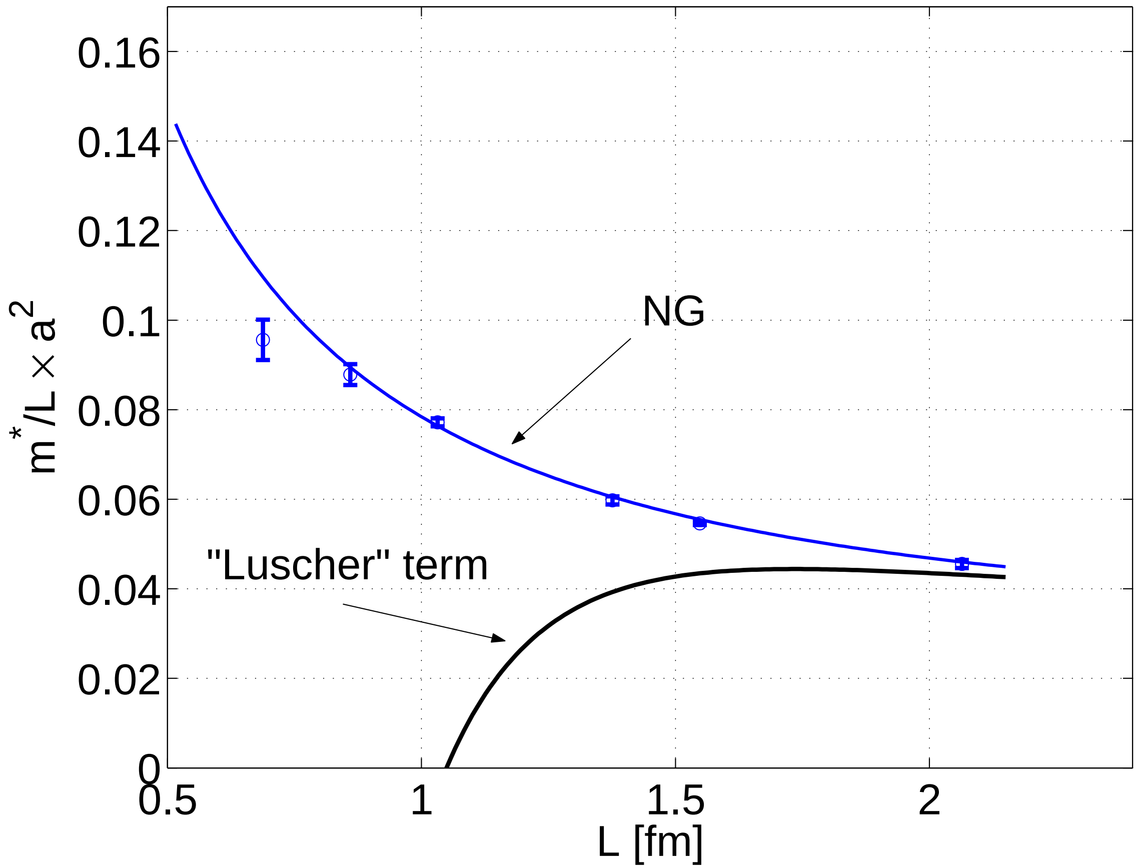
- $E_0 = \sqrt{\underbrace{(\sigma L)^2 - \frac{\pi\sigma}{3}}_{\text{NG}} - \frac{C}{\sqrt{\sigma}L^3}}$

Nambu-Goto is a Very good approximation for the YM closed flux-tube.

Results - comparison with Nambu-Goto

$$E_1 = \sigma L + \frac{23\pi(D-2)}{6L} + \mathcal{O}\left(\frac{1}{L^2}\right)$$

Do $SU(N)$ with $N = 3, 4, 6, 8$ and $a^{-1} \simeq 1.6, 2.6$ 'GeV' $a\sqrt{\sigma} \simeq 0.25, 0.15)$

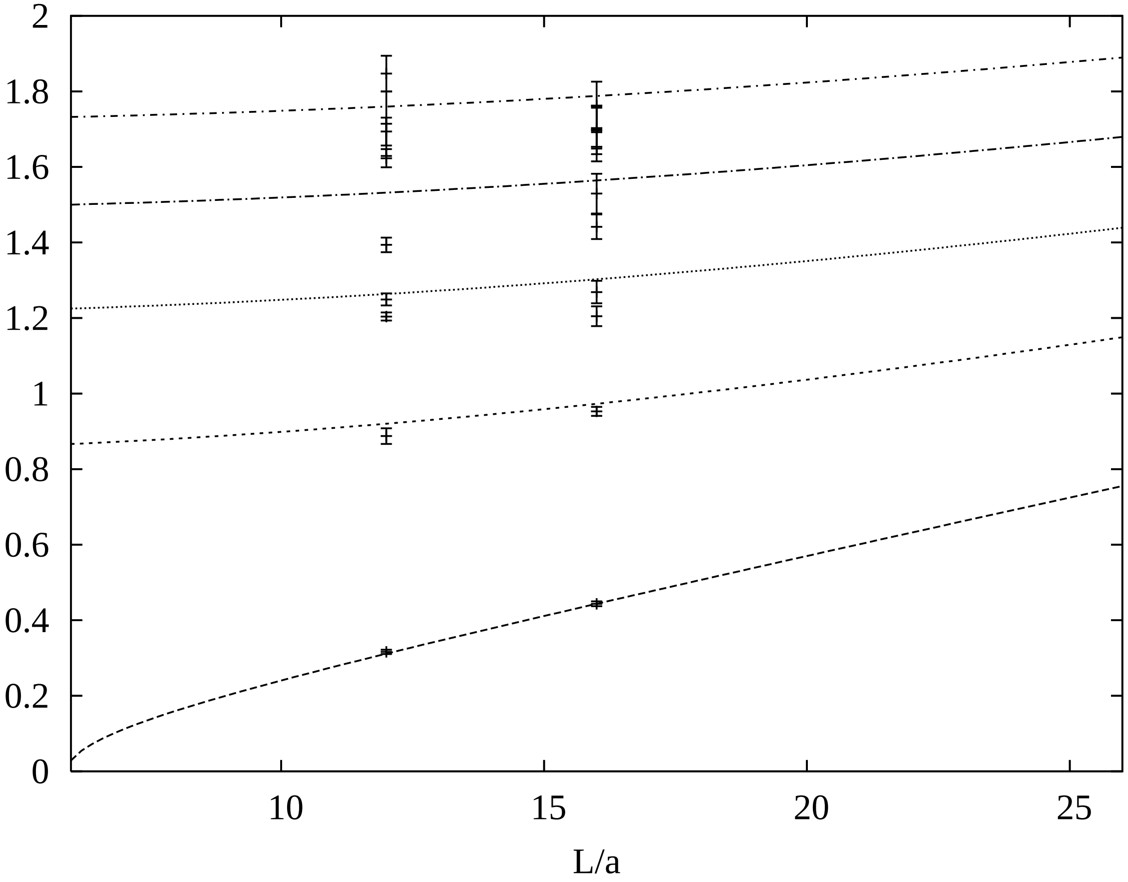


- 1st excited state.
- $SU(6)$, $a^{-1} \simeq 2.6$GeV, $a\sqrt{\sigma} \simeq 0.17$.
- $E_1 = \sqrt{\underbrace{(\sigma L)^2 + \frac{23\pi\sigma}{3}}_{\text{NG}}}$

Nambu-Goto is a Very good approximation for the YM closed flux-tube.

Results - comparison with Nambu-Goto

Preliminary with 118 operators - plan to decrease errors by a factor of $1/3$ or more



- ground state + (1st-4th) excited state.
- $\boldsymbol{SU(3)}$, $a^{-1} \simeq 2.6\text{GeV}$, $a\sqrt{\sigma} \simeq 0.17$.
- $$\boldsymbol{E_n = \sqrt{\underbrace{(\sigma L)^2 + 8\pi\sigma\left(n - \frac{1}{24}\right)}_{\text{NG}}}}$$
- See degeneracies.

Nambu-Goto is a Very good approximation for the YM closed flux-tube.

Results - comparison with Karabali-Kim-Nair

Results for $SU(3,4,6,8)$, and $a^{-1} = 1.6, 2.6$'GeV' encompassed by :

$$\frac{E_0}{\sigma L} = \sqrt{\underbrace{1 - \frac{\pi}{3}\left(\frac{1}{\sqrt{\sigma}L}\right)^2}_{\text{NG}} - 0.2(1)\left(\frac{1}{\sqrt{\sigma}L}\right)^5}$$

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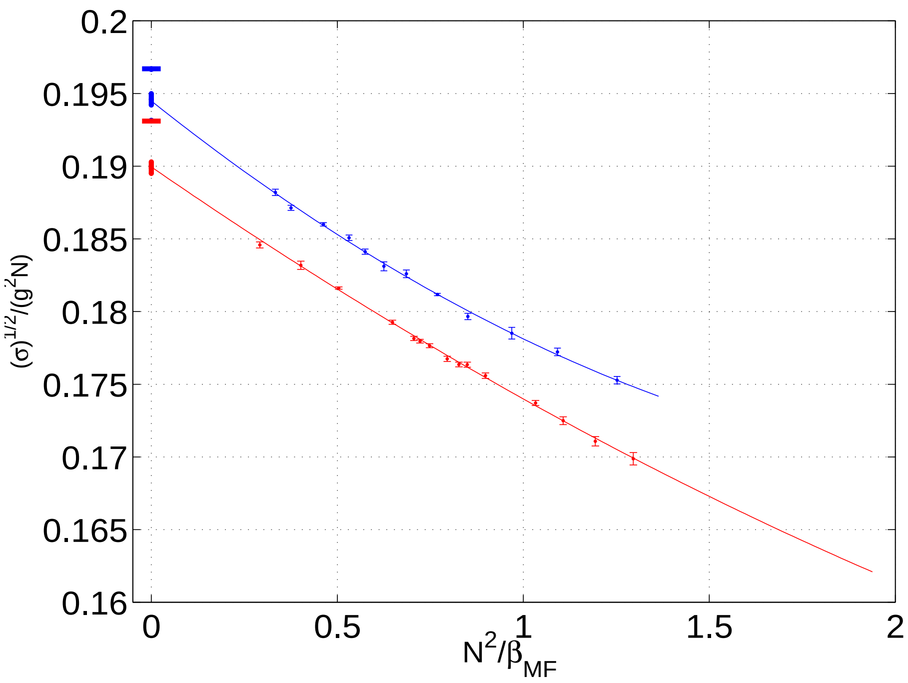
Do continuum limit get $\frac{\sqrt{\sigma}}{g^2 N}$ for $N = 2, 3, 4, 5, 6, 8$



- Red is $SU(4)$, Blue is $SU(6)$
- Y-axis $= \frac{\sqrt{\sigma}}{g^2(a)N} \xrightarrow{a \to 0} \frac{\sqrt{\sigma}}{g^2 N}$
- X-axis $= \frac{1}{2}\, a\, g^2(a) N \xrightarrow{a \to 0} 0$

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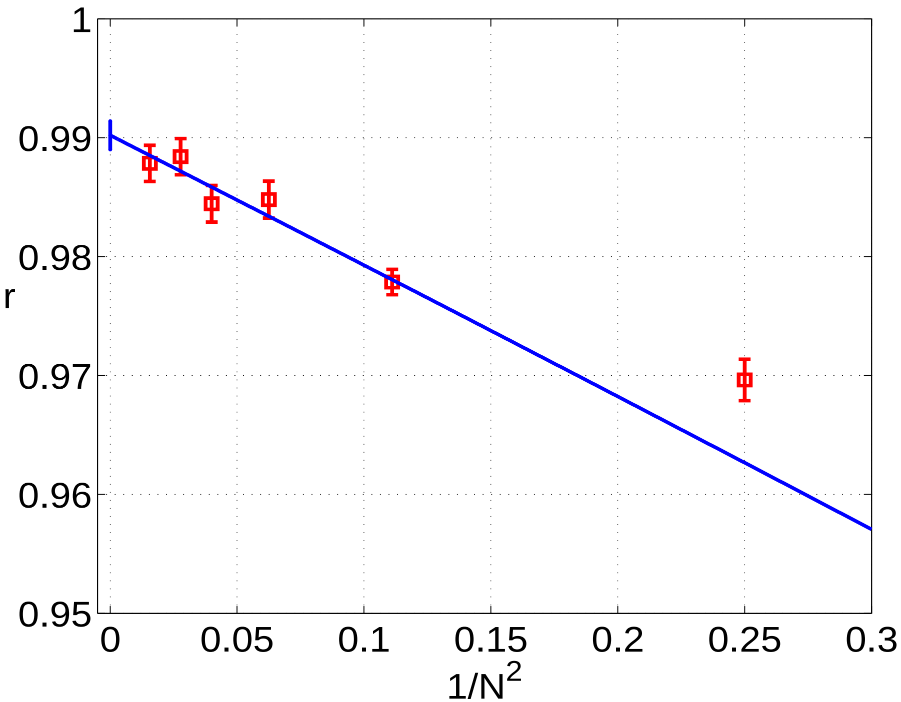
Do large-N limit get $\frac{\sqrt{\sigma}}{g^2 N}$ for $N = \infty$



- $r \equiv \frac{\text{Lattice}}{\text{Karabali−Nair}} \xrightarrow{N\to\infty} \mathbf{0.990 \pm 0.001 - 0.003}$
- $\left(\frac{\sqrt{\sigma}}{g^2 N}\right)_{\text{Lattice}} \xrightarrow{N\to\infty} \mathbf{0.1975 \pm 0.0002 - 0.0005}$
- $\left(\frac{\sqrt{\sigma}}{g^2 N}\right)_{\text{KKN}} \xrightarrow{N\to\infty} \frac{1}{\sqrt{8\pi}} = \mathbf{0.199471\ldots}$

Instability of the closed flux-tube

Results for $SU(3,4,6,8)$, and $a^{-1} = 1.6, 2.6$‘GeV’ encompassed by :

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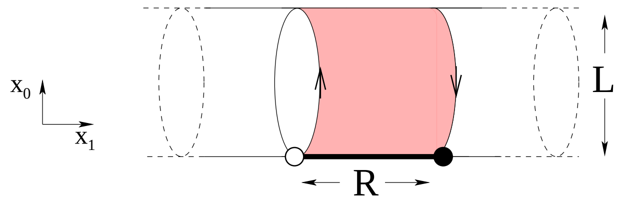

Which means that

$$T = L^{-1}$$

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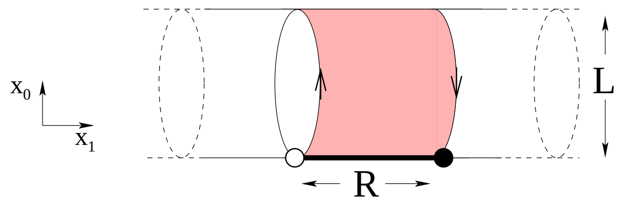


Which means that

$$T = L^{-1}$$

If $E_0^2 < 0$ then $\langle P \rangle \neq 0$? Pisaski and Alvarez '82, Arvis '83

$$\left(\frac{E_0}{\sigma L}\right)^2 = 1 - \left[\sqrt{\frac{\pi(D-2)}{3}}\frac{1}{\sqrt{\sigma}L}\right]^2 = 1 - \left[\sqrt{\frac{\pi(D-2)}{3}}\frac{T}{\sqrt{\sigma}}\right]^2 \leq 0$$

This happens when

$$\frac{L^{-1}}{\sqrt{\sigma}} = \frac{T}{\sqrt{\sigma}} \geq \sqrt{\frac{3}{\pi(D-2)}} \equiv \frac{T_{\text{Hagedorn}}(NG)}{\sqrt{\sigma}} \qquad \text{where} \quad E_{\text{loop}} \to 0$$

The Hagedorn temperature

The Ultimate temperature idea by Hagedorn '65 - pre-dated QCD

pp particle multiplicities $\longrightarrow \rho(E) \sim e^{\# E} \longrightarrow$ hadrons consistent if $T < T_H \simeq 158$ MeV.

Interpret T_H as 2nd-order transition of quark liberation by Cabibo and Parisi '75.

Heuristic description for Hagedorn/deconfinement given by Banks and Rabinovici '79:

Confinement means

$$\left.\begin{array}{l} E(l) = \sigma\, l \\ \rho(l) = \exp\left(+c\, l\right) = \exp\left(+\frac{c}{\sigma}\; E\right) \end{array}\right\} \longrightarrow Z(T) = \sum_E \rho(E)\, e^{-E/T} = \sum_E e^{(c/\sigma - 1/T)E}.$$

So $Z(T = \frac{\sigma}{c}) \longrightarrow \infty$ and long loops proliferate with $m(T = T_H \equiv \frac{\sigma}{c}) = 0$.

Calculating $\rho(E)$ in 2+1 for Nambu-Goto gives :

$$T_H/\sqrt{\sigma} = T_{\mathrm{Hagedorn}}(NG)/\sqrt{\sigma} = \sqrt{\frac{3}{\pi}} \qquad !! \qquad \text{Caselle '05}$$

These arguments lead to 2nd order and ignore interactions between (string-like)Hadrons.

$$\Downarrow$$

Natural at $N = \infty \Rightarrow$ $\boxed{\text{2nd order transition at } N = \infty \text{ ?}}$ No ! Lucini, Teper, Wenger ‘03, Liddle and Teper ‘05

Usual way out: Deconfinement at T_d is not a Hagedorn phenomenon.

At $\boldsymbol{T = T_d}$: $\boxed{\boldsymbol{F(\mathrm{quarks}\,\&\,\mathrm{gluons}) < F(\mathrm{confining}-\mathrm{string}-\mathrm{theory})}}$

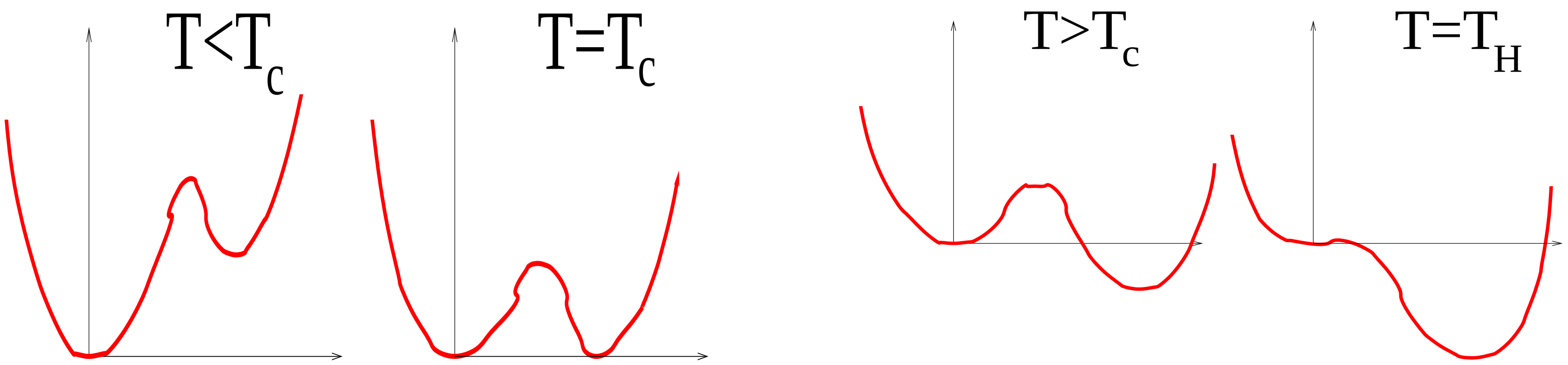
$\Longrightarrow$ the g.s. completely changes and ‘forgets’ that it had a string-description.

Relation between T_d **and** T_H = a dynamical question.

E.g. **analytical calculations at** $N = \infty$ **on a sphere** :

Aharony et al.
'03-'05

- Get $V(\langle P \rangle)$ as a perturbative planar series: 1st order at T_c, spinodal point at T_H



Aim to check this picture on the (infinite-volume) lattice.

Seeking T_H on the lattice - strategy

Probe the metastable confined vacuum for $T > T_d$, by measuring energy of the close-flux tubes/Polyakov loops.

Choose large values of N because

- **Theory**: It is at large-N that the string description becomes more natural.
- **Practice**: suppressed tunnellings

$$P(T_d) \propto \exp\{-2\sigma_{cd}A/T_d\} \sim \exp(-\#N^2).$$

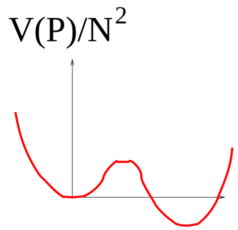
Lucini, Teper and Wenger '05

Cannot get to T_H itself because tunnelling to true vacuum.

Can hope to - go high enough in T to allow an extrapolation to $m(T_H) = 0$.

How should we extrapolate ?

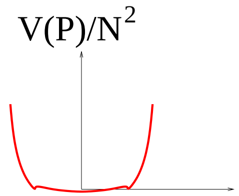
If $T_H = T_{\text{spinodal}}$:



- Stay in metastable phase only far from T_H.
- Before that, fluctuations are Gaussian.

$$\Rightarrow \qquad m \sim (T_H - T)^{1/2}$$

If $T_H < T_{\text{spinodal}}$:

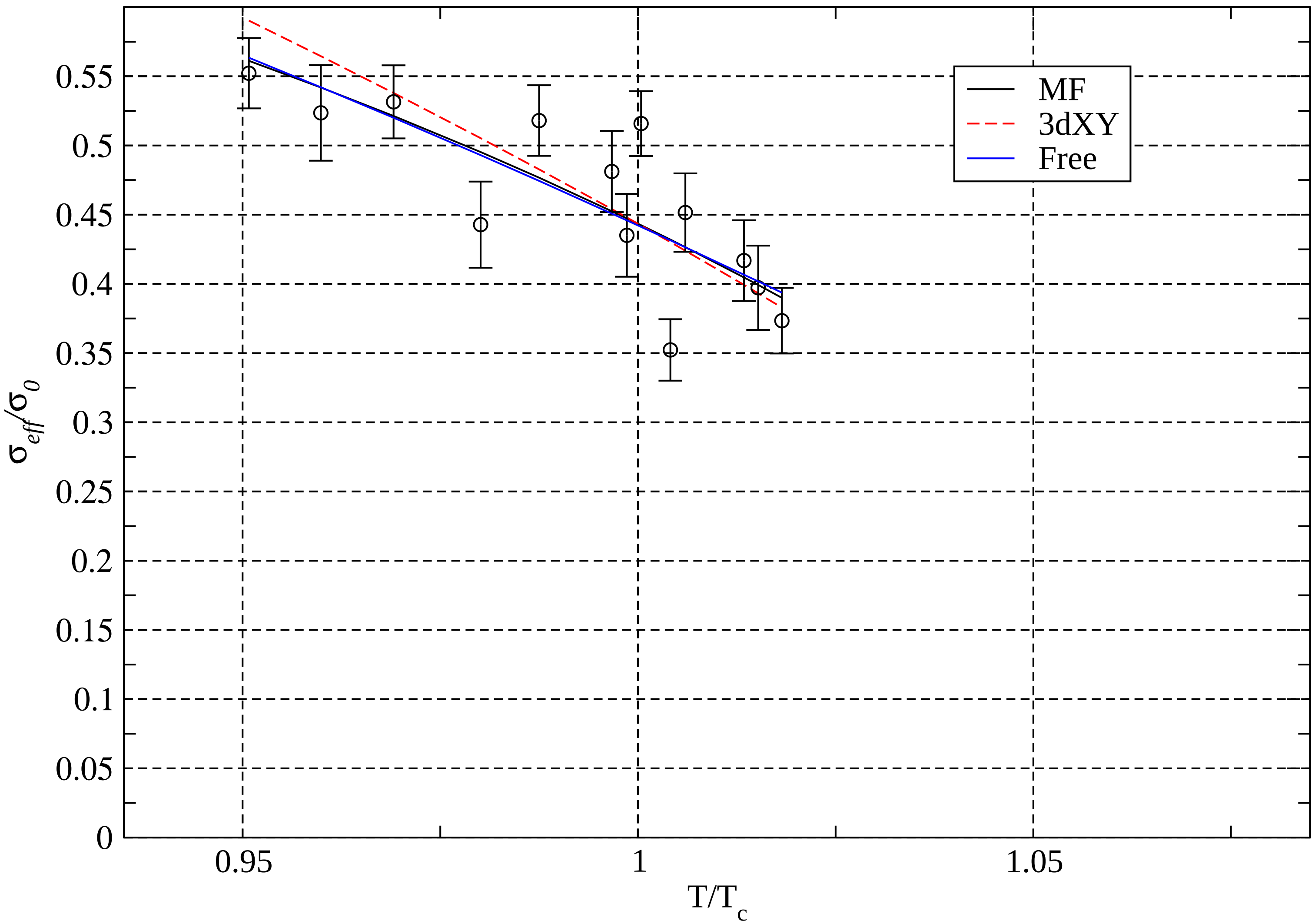


- Dictated by Universality $\longrightarrow$ 3DXY
- Fluctuations $= 1/N^2$: affect vicinity of T_H.

Vicinity of T_H	$\Rightarrow$	$m \sim (T_H - T)^{0.6715}$	for the 3DXY
Further away from T_H	$\Rightarrow$	$m \sim (T_H - T)^{1/2}$	for the MF

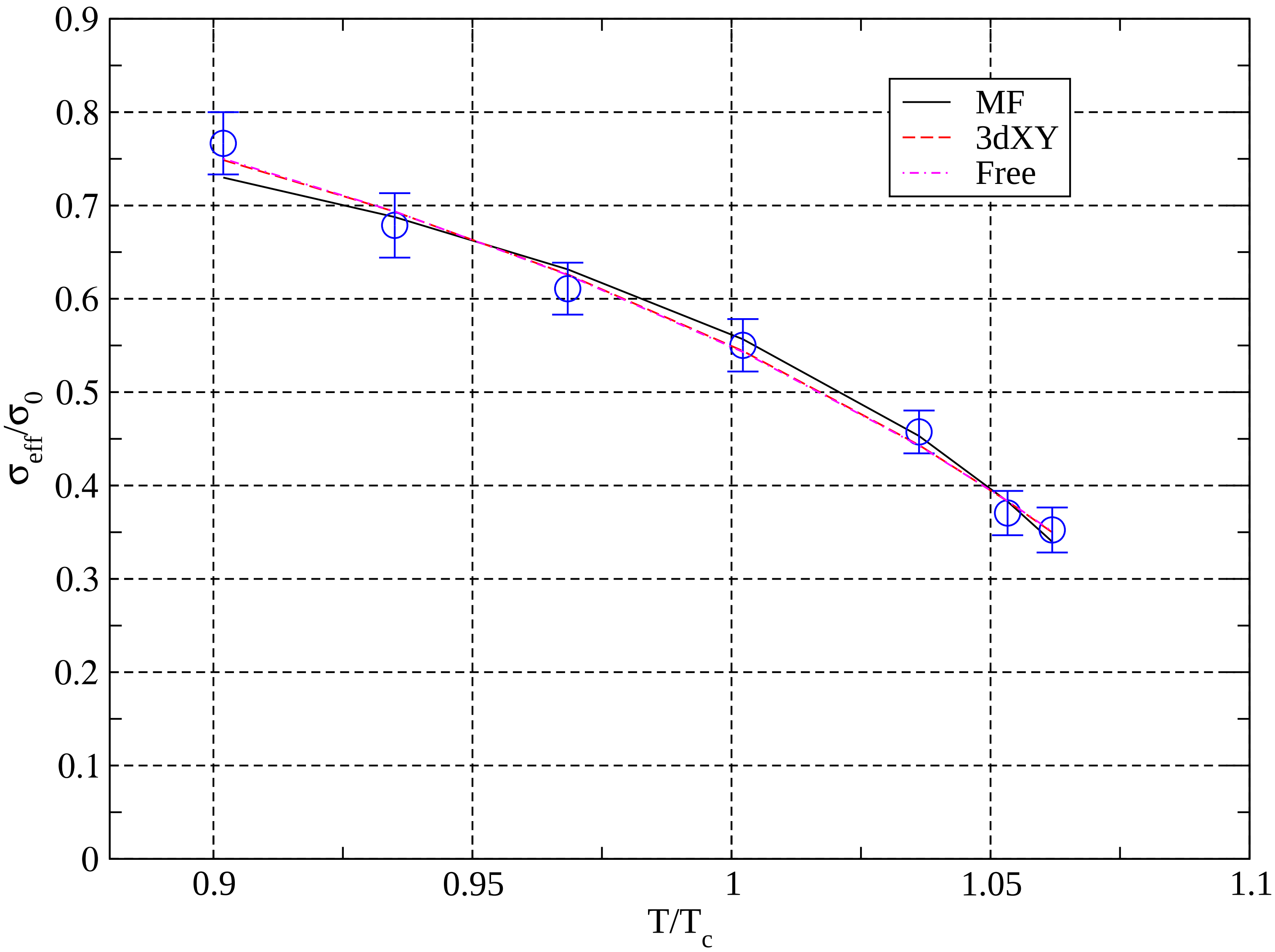
Do $SU(8, 10, 12)$ on $a^{-1} \simeq 1.4$GeV, $a\sqrt{\sigma} \simeq 0.35$, and get

$$\boldsymbol{\sigma}_{\rm eff} \equiv \boldsymbol{m}_{\rm loop}/\boldsymbol{T}^{-1}$$



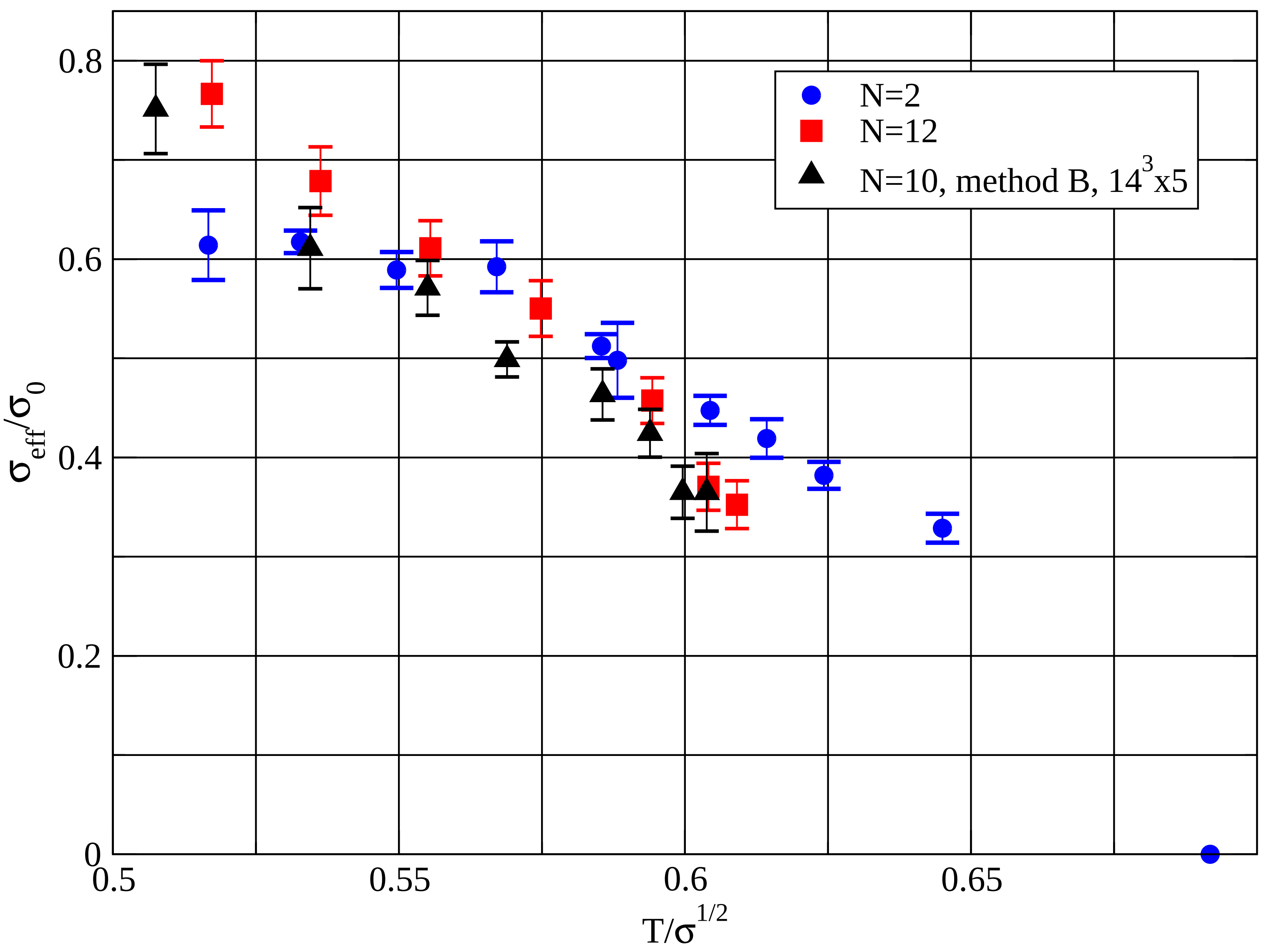
$$\boldsymbol{SU(8)}$$

$$\boldsymbol{\sigma}_{\rm eff} \equiv \boldsymbol{m}_{\rm loop}/\boldsymbol{T}^{-1}$$



$$\boldsymbol{SU(12)}$$

$$\sigma_{\rm eff}/\sigma_0 \equiv m_{\rm loop}/T^{-1}$$



$SU(12,10)$ vs. $SU(2)$. Note $\frac{T_{\rm Hagedorn}(NG)}{\sqrt{\sigma}} = 0.691\ldots$

Seeking T_H on the lattice - results

The true transition is strongly first order and

$$\sigma_{\rm eff}(T_d) \simeq \frac{1}{2}\sigma \text{ for } N = 8, 10, 12.$$

Most statistically reliable result for $SU(12)$ loop mass vanishes at

$$T_H/T_d = 1.116(9) - 1.092(6) \qquad ; \qquad T_H/\sqrt{\sigma} = 0.63 - 0.64$$

Interpretations for $T_H/\sqrt{\sigma} < T_{\rm Hagedorn}(NG)/\sqrt{\sigma} = 0.691$:

- Dalley '05 : Eff. central charge $c_{\rm eff} \equiv$ effective # d.o.f. instead $D-2$ oscillations.

$$\frac{T_H}{\sqrt{\sigma}} \equiv \sqrt{\frac{3}{c_{\rm eff}\,\pi}} \longrightarrow c_{\rm eff} \simeq 2.4 \overset{D=4}{>} D-2 \rightarrow \text{Massive modes in the flux-tube ?}$$

- Back to $2+1$: when $E_0 = \sigma T^{-1}\sqrt{1 - \frac{\pi}{3}\left(\frac{T}{\sqrt{\sigma}}\right)^2 - 0.2\left(\frac{T}{\sqrt{\sigma}}\right)^5} \overset{?}{=} 0$

$$\frac{T_d^{2+1}}{\sqrt{\sigma}} \simeq 0.89 < \frac{T}{\sqrt{\sigma}} \simeq 0.91 < \frac{T_{NG}^{2+1}}{\sqrt{\sigma}} \simeq 0.98 \longrightarrow \text{same pattern as in } 3+1 \text{ ?}$$

Summary

Tested (Karabali-Kim-Nair)$_{N=\infty}$ while attempting to control systematic errors

Only mentioned

- Excited state contamination.
- $O(g^4)$ in continuum limit.
- $O(1/N^4)$ in large-N limit.

Discussed

- Beyond the Luescher term.

$\Rightarrow$ **KKN is amazingly close ($\sim 1\%$)**
But
Not exact ($\sim$ 6 sigma)

Calculated flux-tube masses as a function of their length in $N = 3, 4, 6, 8$

$$E_0 = \sigma L \sqrt{1 - \frac{\pi}{3}\left(\frac{1}{\sqrt{\sigma}L}\right)^2 - 0.2(1)\left(\frac{1}{\sqrt{\sigma}L}\right)^5}$$

$\Rightarrow$ **Nambu-Goto is a very good approximation for the flux-tube**

Looked for signs of T_H in $N = 8, 10, 12 \quad (D = 3+1)$: $\Rightarrow$ $\mathbf{T_H \simeq 1.1 T_d}$

Many future prospects

$T = 0, \quad 2+1$:

- k-strings.
- Non Nambu-Goto behavior : negative parity, glueball mixing.
- Glueball spectra to check Minic et al. a lá Karabali-Nair.

$T = 0, \quad 3+1$:

- String & Glueball spectra are richer (nontrivial angular momentum).
- Quarkonia at large-N. Neuberger and Narayanan, recent years

$T > 0$:

- Large-N on a sphere (currently $2+1$) BB and Wheater
- Eigenvalue spectra of Polyakov loops Aharony et al, Pisarski et al.