Spring 1999
Qualifying Examination

Quantum Mechanics

Clebsch-Gordan Coefficients:

$$< j_1 1; m_1 0 | j_1 1; j_1 m_1 > \;=\; \frac{m_1}{\sqrt{j_1(j_1+1)}}$$

$$< j_1 2; m_1 0 | j_1 2; j_1 m_1 > \;=\; \frac{3m_1^2 - j_1(j_1+1)}{\sqrt{(2j_1-1)j_1(j_1+1)(2j_1+3)}}$$

Spherical Harmonics:

$$Y_{2,0}(\theta,\phi) = \sqrt{\frac{5}{4\pi}}\left(\frac{3}{2}\cos^2\theta - \frac{1}{2}\right).$$

Pauli matrices:

$$\boldsymbol{\sigma_x} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$\boldsymbol{\sigma_y} = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

$$\boldsymbol{\sigma_z} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

1. a) [10 pts.] An electron and a positron interact non-relativistically via the Coulomb force. Write a spin wavefunction for the system (*i.e.* omit orbital angular momentum and radial excitations) in terms of the elementary spins s and projections m_s of the two particles. Is the Hamiltonian diagonal? What is the degeneracy of states?

   b) [20 pts.] The magnetic moment of the electron is $\boldsymbol{\mu} = -g\mu_B \mathbf{s}$, where g and μ_B are constants. Add a perturbation term to the Hamiltonian in part a) accounting for the interaction of the magnetic moments (you may omit units). Show that this perturbation is diagonal in a basis of total spin $\mathbf{S} = \mathbf{s}_1 + \mathbf{s}_2$. Give the relative sizes of the diagonal matrix elements of the perturbation and the degeneracies for the states of the perturbed system.

2. a) [5 pts] Isospin is a quantum number analogous to spin. In nature it is found that protons and neutrons have nearly identical masses and can be treated as the two isospin projections of an isospin-$\frac{1}{2}$ "nucleon." Define isospin Pauli matrices $\boldsymbol{\tau}$ and $\mathbf{t} = \frac{1}{2}\boldsymbol{\tau}$ with the required properties and write down the Cartesian components of $\boldsymbol{\tau}$. Evaluate the isospin z-projections of a neutron and of a proton with the operator you define.

b) [10 pts] Defining a total isospin operator $\mathbf{T}$ in terms of the $\mathbf{t}_i$ acting on the ith nucleon, in a nucleus of Z protons and N neutrons, what values can the total isospin and the isospin projection take on? If the strong interaction Hamiltonian H_N is independent of the isospin projections of individual nucleons, what is the degeneracy of a state with total isospin T (when N and Z are not constrained)?

c) [10 pts] Justify that the Coulomb interaction may be written in the form

$$H_C = e^2 \sum_{i<j} \left(\frac{1}{2} \pm t_z(i) \right) \left(\frac{1}{2} \pm t_z(j) \right) \frac{1}{r_{ij}}.$$

with the choice of sign depending on your definition of the relevant operators.

d) [15 pts] By adding and subtracting the term $\frac{1}{3}\mathbf{t}(i)\cdot\mathbf{t}(j)$ into the above expression and grouping terms (or otherwise) show that the Coulomb part $H_C \equiv H_C^0 + H_C^1 + H_C^2$ contains the sum of a scalar, the z-component of a vector and the zeroth component of an irreducible spherical tensor operator of rank 2.

3. a) [20 pts] A quantum-mechanical system (a nucleus) forms eigenstates of a Hamiltonian H_N, which is of rank 0 in the space defined by $\mathbf{T}$. Additional (Coulomb) contributions of rank k, projection 0, H_C^k ($0 \leq k \leq 2$), are a perturbation on H_N. Write the energy perturbations to states characterized by $|\alpha T M_T >$ (α stands for other quantum numbers). Apply the Wigner-Eckart theorem to the perturbation expression and show that the perturbations can only be of quadratic order in M_T, irrespective of the value of T. [This result is known as the Isobaric Multiplet Mass Equation and was first derived by Wigner in 1957.]

b) [10 pts] From the structure of the perturbation matrix elements, what is the physical significance of the fact that only the zeroth components of the (3) tensor operators emerge in the treatment of the Coulomb force?