Spring 2005 Qualifying Examination - Electromagnetism.

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1. [**55** points total] **(Dipolar Radiation)**

A. [**20** points] The vector potential generated by a current $\boldsymbol{J}(\boldsymbol{x}', t')$ in the Lorentz gauge is:

$$\boldsymbol{A}(\boldsymbol{x}, t) = \frac{1}{4\pi\epsilon_0 c^2}\int dt' \int d^3x' \frac{\boldsymbol{J}(\boldsymbol{x}', t')}{|\boldsymbol{x}' - \boldsymbol{x}|}\delta(t' - t + \frac{|\boldsymbol{x}' - \boldsymbol{x}|}{c}).$$

Show that, in the radiation zone ($k|\boldsymbol{x}| \gg 1 \gg k|\boldsymbol{x}'|$), the vector potential generated by a dipole $\boldsymbol{p}(t') = \mathrm{Re}\left[\boldsymbol{p}_0 e^{-i\omega t'}\right]$ located at the origin is:

$$\boldsymbol{A}(\boldsymbol{x}, t) = \mathrm{Im}\left[\frac{k\boldsymbol{p}_0 e^{-i\omega t}}{4\pi\epsilon_0 c}\frac{e^{ik|\boldsymbol{x}|}}{|\boldsymbol{x}|}\right]$$

with $k = \omega/c$. HINT: if $k|\boldsymbol{x}| \gg 1 \gg k|\boldsymbol{x}'|$ then $|\boldsymbol{x}' - \boldsymbol{x}| \approx |\boldsymbol{x}|$.

B. [**35** points] A dipole $(0, 0, p_0)\cos\omega t$ is located at the origin:

i [**20** points]Show that the magnetic field in the radiation zone is:

$$\boldsymbol{B}(\boldsymbol{x}, t) = \mathrm{Re}\left[\frac{k^2}{4\pi\epsilon_0 c}(\boldsymbol{n} \times \boldsymbol{p}_0)\frac{e^{i(k|\boldsymbol{x}| - \omega t)}}{|\boldsymbol{x}|}\right]$$

and calculate the electric field. Is the polarization of the radiation circular, plane, or of another kind? Justify your answer.

ii [**10** points] Calculate the Poynting vector and show that the radiated power is:

$$\frac{dP}{d\Omega} = \frac{c}{32\pi^2\epsilon_0}k^4|\boldsymbol{p}_0|^2 f(\theta).$$

Indicate what is the function $f(\theta)$.

iii [**5** points] Consider an electron oscillating with a period $T = 10^{-6}$ seconds and an amplitude of 10^{-4} m. Calculate the power (order of magnitude) received by a 1-cm^2 plate located 1 meter away from the dipole at $\theta = 90$ degrees.($\frac{1}{4\pi\epsilon_0} \approx 9 \times 10^9$ N m^2/C^2.)

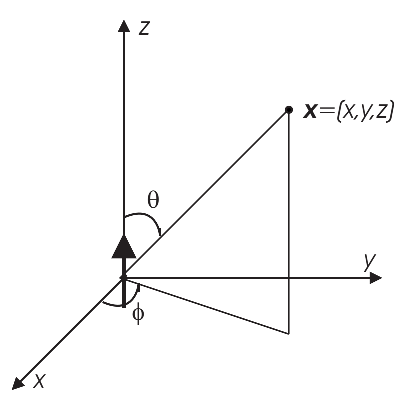

FIG. 1: *Dipole in z-direction.*

2. [**45** points total] **(Faraday Effect)**

A plane wave with frequency ω travels in a medium filled with molecules that can be considered as harmonic oscillators with charge q, mass m, and spring constant κ.

A. [**20** points] Show that, neglecting the effect of the magnetic field of the wave on the molecules' oscillations, the index of refraction is:

$$n = \sqrt{1 + \frac{Nq^2}{\epsilon_0 m(\frac{\kappa}{m} - \omega^2)}}$$

where N is the number of molecules per unit volume.
HINT: write the equations of motion for the harmonic oscillator in the presence of the electric field of the wave; then, use it to calculate the polarization.

B. [**20** points]Assume now that a static magnetic field in the direction of propagation of the wave is present in the medium. Show that the two states of circular polarization (R/L) of the wave have a different index of refraction given by:

$$n = \sqrt{1 + \frac{Nq^2}{\epsilon_0 m(\frac{\kappa}{m} - \omega^2 \pm \omega\frac{qB}{m})}}.$$

where the upper and lower signs corresponds to the two states of circular polarization of the wave. As in point **2.A** ignore the magnetic field generated by the wave.

C. [**5** points] Explain what would happen to an electro-magnetic wave initially plane-polarized as it travels through a medium as described in point **2.B** above.