Fall 2005 Qualifying Examination - Electromagnetism.

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1. [50 points total] Retarded potentials

The retarded potentials

$$V(\boldsymbol{x},t)=\frac{1}{4\pi\epsilon_0}\int\frac{\rho(\boldsymbol{x}',t_r)}{|\boldsymbol{x}-\boldsymbol{x}'|}d^3x' \quad \boldsymbol{A}(\boldsymbol{x},t)=\frac{\mu_0}{4\pi}\int\frac{\boldsymbol{J}(\boldsymbol{x}',t_r)}{|\boldsymbol{x}-\boldsymbol{x}'|}d^3x' \tag{1}$$

where $t_r=t-\frac{|\boldsymbol{x}-\boldsymbol{x}'|}{c}$, are solutions to the wave equation (written in the Lorentz gauge):

$$\Box V(\boldsymbol{x},t)=-\frac{\rho(\boldsymbol{x},t)}{\epsilon_0} \quad \Box\boldsymbol{A}(\boldsymbol{x},t)=-\mu_0\boldsymbol{J}(\boldsymbol{x},t). \tag{2}$$

As a consequence, for example, the potentials due to a point particle with charge q moving with constant velocity $\boldsymbol{v}$ are (Liénard-Wiechert potentials):

$$V(\boldsymbol{x},t)=\frac{q}{4\pi\epsilon_0}\frac{1}{|\boldsymbol{\mathfrak{r}}|-\boldsymbol{\mathfrak{r}}\cdot\boldsymbol{v}/c} \quad \boldsymbol{A}(\boldsymbol{x},t)=\frac{\boldsymbol{v}}{c^2}V(\boldsymbol{x},t) \tag{3}$$

where $\boldsymbol{\mathfrak{r}}$ is the displacement vector from the retarded position to the point where the potential is calculated.

A. [**15** points] A bar is moving at constant speed u as shown in the figure. Its length in the lab is ℓ, it has negligible diameter and has charge Q uniformly distributed over its length. Use Eqs. 1 to calculate the potentials at point $(x_0,0,0)$ as a function of time. Assume the bar is to the right of $(x_0,0,0)$ as shown in the figure.

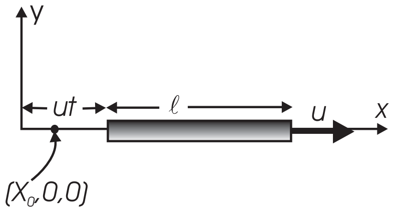

FIG. 1: *Uniformly charged bar moving at constant velocity, u.*

B. [**10** points]Show that your expressions agree with the Liénard-Wiechert potentials of Eqs. 3 in the limit $\ell\to 0$ with u/c fixed.

C. [**15** points]For the $\ell\to 0$ limit calculate the electric and magnetic fields as a function of time at the origin. Start with the potentials you calculated under [**1.B.**] or with Eqs. 3.

D. [**10** points]Re-calculate the fields of **1.C.** but here start with Coulomb's law in the rest frame of the charge and make the appropriate Lorentz transformation.

2. [**50** points total] **Magnetic field from currents.**

A cable consists of a material of magnetic permeability $\mu = \mu_0$ with the shape of an infinitely long cylinder of radius R, covered by two infinitesimally thin conducting sheets as shown in the figure. The two conducting sheets are insulated from each other and carry surface current densities $\boldsymbol{\sigma} = \sigma_0\hat{\boldsymbol{z}}$ (top sheet, $0 \le \phi \le \pi$) and $\boldsymbol{\sigma} = -\sigma_0\hat{\boldsymbol{z}}$ (bottom sheet, $\pi \le \phi \le 2\pi$.)

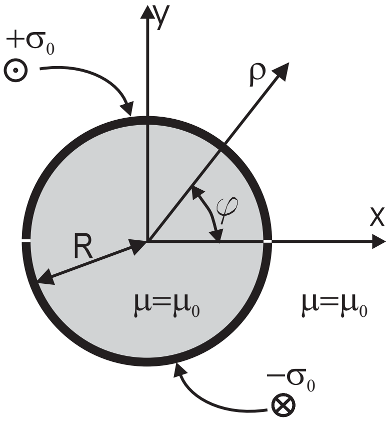

FIG. 2: *Cylindrical cable consisting of a core of radius R and two conducting sheets carrying opposing currents.*

A. [**15** points] Given that the volume current is zero show that one can use a scalar potential, V, such that the field can be calculated everywhere as $\boldsymbol{H} = \boldsymbol{\nabla} V$. Using Maxwell's equations show that $\nabla^2 V = 0$ and write down the general form of the solutions (inside and outside the cable) in cylindrical coordinates and write down the boundary conditions that should be used.

B. [**10** points]Use qualitative arguments to sketch the field lines.

C. [**15** points] Show that the fields outside the cable, $\boldsymbol{B}^{\mathrm{ext}}$, and inside the cable, $\boldsymbol{B}^{\mathrm{in}}$, are:

$$\boldsymbol{B}^{\mathrm{ext}} = A^{\mathrm{ext}} \sum_{n=1,3,5,\ldots} \left(\frac{R}{\rho}\right)^{n+1} \frac{1}{n}(-\hat{\boldsymbol{\rho}}\cos n\phi - \hat{\boldsymbol{\phi}}\sin n\phi)$$

$$\boldsymbol{B}^{\mathrm{int}} = A^{\mathrm{int}} \sum_{n=1,3,5,\ldots} \left(\frac{\rho}{R}\right)^{n-1} \frac{1}{n}(\hat{\boldsymbol{\rho}}\cos n\phi - \hat{\boldsymbol{\phi}}\sin n\phi)$$

and determine the constants A^{ext}, A^{int} in terms of the characteristics of the cable.

D. [**10** points] At large distances from the cable the field is given by:

$$B \to \frac{C}{\rho^2}(-\hat{\boldsymbol{\rho}}\cos\phi + \hat{\boldsymbol{\phi}}\sin\phi)$$

which coincides with the field given by two wires located at $x = 0, y = \pm d/2$ carrying currents $\pm I\hat{z}$. Find a current I and distance d that would produce the same field as the cable of this problem at large distances from the cable.