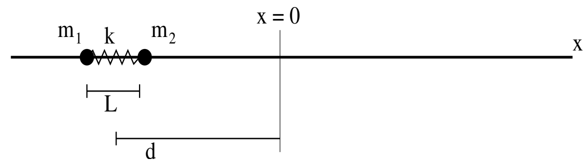
Autumn 2005 Qualifying Examination - Classical Mechanics.

In this question, you will solve a sequence of related problems. By the end, you may be able to uncover a remarkable dynamical property of stars.

The problem looks long, but many of the required answers are qualitative.

I) **Effect of an External Force on particles connected by a spring.** [25 points total]

Two point masses, with masses m_1, m_2, are connected by a spring of spring constant k and equilibrium length L. The masses are constrained to move (without friction) only on the x axis.



Before $t = 0$, the masses are at rest, with $x_1 = -d - L/2$ and $x_2 = -d + L/2$ (see figure.)

External forces act on the masses for a time interval of length T. For $0 < t < T$, the force on mass 1 is $F_0\hat{x}$, while the force on mass 2 is $4F_0\hat{x}$, where F_0 is a positive constant. At all other times the forces are zero.

[5 points] A) warmup: if $k = 0$ (no spring), what are $x_1(t)$ and $x_2(t)$ for $t > 0$?

[7 points] B) warmup: if $k = \infty$ (rigid rod), what are $x_1(t)$ and $x_2(t)$ for $t > 0$?

[13 points] C) Now suppose k is finite but very small. Also take $m_1 = m_2 = m$. Approximately, what are $x_1(t)$ and $x_2(t)$ for $t > T$? *[Hint: use the approximation that k is very small to break the problem into two simple parts.]*

II) **General Questions** [25 points total]

Now we add particle 3 into the system.

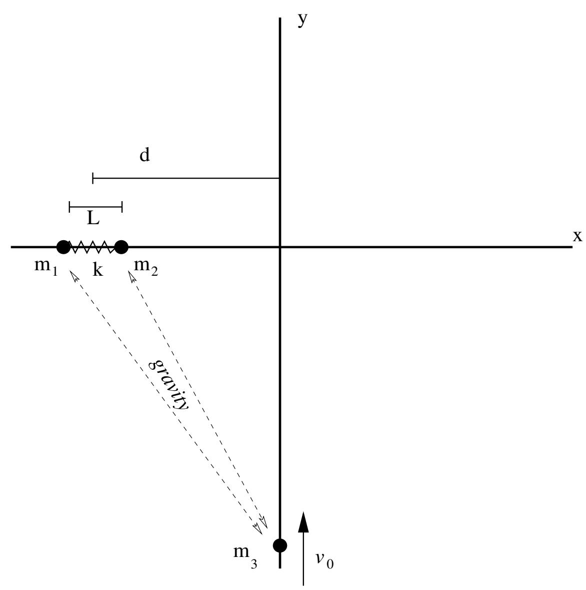
Particle 3 is *constrained* to move (without friction) only on the y-axis; particles 1 and 2 are similarly *constrained* to move (without friction) only on the x-axis.

Particle 3 interacts with particles 1 and 2 **via ordinary gravity**:

$$F_{13} = -G\frac{m_1 m_3}{r^2} \ , \ F_{23} = -G\frac{m_2 m_3}{r^2}$$

where G is Newton's constant.

Particles 1 and 2 are still bound by a spring. *[Ignore all gravitational interactions between particles 1 and 2; they are small compared to the forces exerted by the spring.]*



[7 points] A) **Write the exact Lagrangian** for this system, as a function of x_1, x_2, y_3.

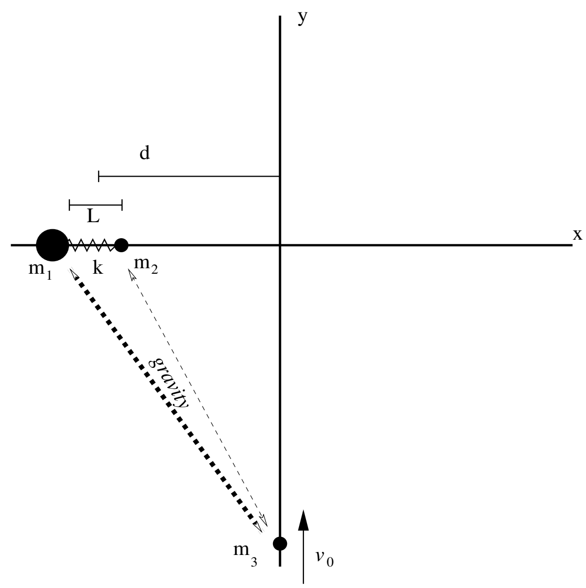
[6 points] B) **Write the exact equations of motion** for object 2.

[12 points] C) Are there any **conserved quantities** in this system, and if so, what are they? *[Hint: The motion is constrained.]* Be sure that you justify your answer.

III) **A Special Case** The forces are as in part II. [25 points total]

Now let $m_1 \gg m_2 = m_3$.
In this case x_1 is nearly constant, but x_2 may oscillate.

At $t = -\infty$, particles 1 and 2 are at rest with $x_1 = -d - L/2$, $x_2 = -d + L/2$, where $L \ll d$.
At $t = -\infty$, particle 3 is at $y = -\infty$ and has velocity $v_0\hat{y}$.



[8 points] A) The motion of mass 1 is small, as is the force of particle 2 on particle 3; but the force of mass 1 on mass 3 is large and causes a large $\ddot{y}_3$.
In the approximation that you **only** compute the effect of mass 1 on the motion of mass 3, **show that** the maximum value of $\dot{y}_3(t)$ is

$$\text{Maximum}\Big[\dot{y}_3(t)\Big] \approx \sqrt{v_0^2 + \frac{2Gm_1}{d}}$$

[8 points] B) Now include the gravitational interactions between masses 2 and 3. This is a very small effect on particle 3 (on which the dominant force is that from particle 1.) However, gravity has an important effect on particle 2: it causes x_2 to oscillate. (Meanwhile, $\dot{x}_1$ is very small; it can be ignored.)
After particle 3 has moved well past $y_3 = 0$, particle 2 will generally continue to oscillate. **For very small k, estimate [do not calculate exactly] the energy** stored in the oscillations of x_2 after particle 3 has gone by. You may assume the amplitude of oscillation is small compared to L.

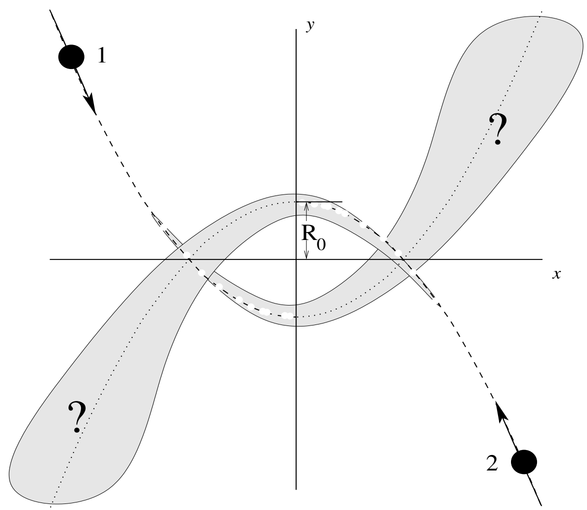
[9 points] C) Suppose we take $v_0 \to 0$ (so that the initial motion of particle 3 is extremely slow, though not quite exactly zero.) In this limit, including all interactions (but with k still small), **make a qualitative graph** of $\dot{y}_3(t)$ versus t. Make sure your graph captures both the early-time and late-time behavior of $\dot{y}_3(t)$. Briefly justify your answer [two or three sentences.]

IV) **Passing stars.** [25 points total]

Two identical stars — spheres of gaseous material of radius R_s and mass M — approach each other along *initially parabolic orbits.* In the center-of-mass system, where $x_1 = -x_2$, $y_1 = -y_2$, their paths for t large and negative are

$$y_1 = x_1^2/r_0 - R_0 \ ; \ y_2 = -x_2^2/r_0 + R_0$$

where r_0 and R_0 are the constants which define the particular orbit. Both r_0 and R_0 are several times larger than R_s.



[10 points] A) In what crucial and remarkable way does the motion differ from the corresponding Newtonian two-body problem with point masses?

[8 points] B) Is the condition that the orbits be parabolic necessary for the feature that you have identified as "remarkable" in IVA? Why or why not?

[7 points] C) Without calculating anything, and without trying to be precise, **make a rough plot**, in the x, y plane, of the motion of star number 1. Make sure the "remarkable" feature is clearly evident.

D) Extra Credit: Can you suggest any observable consequence that this phenomenon might have? [Note: in our region of the galaxy, stars almost never pass at such close range.]