Fall 2004 Qualifying Examination - Classical Mechanics

IMPORTANT: Answer BOTH Questions

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1. [**50** points total] (**Normal Modes and Driven Systems**)

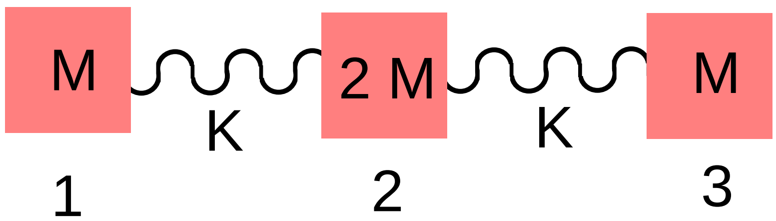

Figure 1: *Three masses joined by two springs.*

A system is comprised of three masses joined by two springs, as shown in the figure. The two springs have equal spring constants. The outer masses have mass M, while the middle mass has mass $2M$. The system is confined to move in one dimension. Let η_i denote the displacement of the i'th mass from its equilibrium position.

A. [**10** points] Find the Lagrangian for this system.

B. [**10** points] Find the Euler Lagrange equations of motion, and show that they can be written as

$$\mathbf{T}\ddot{\boldsymbol{\eta}} + \mathbf{V}\boldsymbol{\eta} \;=\; 0 \;,$$

where $\mathbf{T}$ and $\mathbf{V}$ are matrices and $\boldsymbol{\eta}$ is a column vector. Find $\mathbf{T}$ and $\mathbf{V}$.

C. [**10** points] Find the normal-modes of this isolated system and their frequencies, i.e. find the eigenvalues ω_j, and eigenvectors $\mathbf{a}_j$ of this system.

D. [**10** points] For an arbitrary system with non-degenerate eigenvalues (such as this) show that the eigenvectors can be normalized so that $\mathbf{a}_i^T \; \mathbf{T} \; \mathbf{a}_j \;=\; \delta_{ij}$, where $\mathbf{T}$ is defined as in part B.

Consider the situation where this isolated system is initially at rest, all masses are in their equilibrium positions. Starting at time $t = 0$ a force is applied to mass 1 of the form

$$F(t) \;=\; F_0 \; \cos\left(\Omega \; t\right) \;,$$

E. [**10** points] By writing the displacement $\boldsymbol{\eta}(t)$ in terms of the eigenvectors found above, $\boldsymbol{\eta}(t) \;=\; \alpha_1(t)\;\mathbf{a}_1 \;+\; \alpha_2(t)\;\mathbf{a}_2 \;+\; \alpha_3(t)\;\mathbf{a}_3$, find the displacement of mass 3 as a function of time.

2. [**50** points total] (**Coordinate Systems and the Foucault Pendulum**)

An observer stationed at a point in a coordinate system M whose origin is at $\mathcal{O}$ and with Cartesian unit-vectors $\hat{\mathbf{e}}_x$, $\hat{\mathbf{e}}_y$, and $\hat{\mathbf{e}}_z$ observes a vector $\mathbf{A}$, and its time-derivative $\left.\frac{d\mathbf{A}}{dt}\right|_M$

$$\mathbf{A} = A_1\hat{\mathbf{e}}_x + A_2\hat{\mathbf{e}}_y + A_3\hat{\mathbf{e}}_z \ , \quad \left.\frac{d\mathbf{A}}{dt}\right|_M = \frac{dA_1}{dt}\hat{\mathbf{e}}_x + \frac{dA_2}{dt}\hat{\mathbf{e}}_y + \frac{dA_3}{dt}\hat{\mathbf{e}}_z$$

Subsequently, the observer notices that M is rotating with respect to a coordinate system F that is fixed in space, and whose origin coincides with $\mathcal{O}$.

A. [**12** points] Show that the time-derivative of $\mathbf{A}$ as measured in the fixed coordinate system F, $\left.\frac{d\mathbf{A}}{dt}\right|_F$ can be written as

$$\left.\frac{d\mathbf{A}}{dt}\right|_F = \left.\frac{d\mathbf{A}}{dt}\right|_M + \boldsymbol{\omega}\times\mathbf{A}$$

B. [**12** points] Show that the acceleration of a particle measured in F is related to that measured in M by

$$\left.\frac{d^2\mathbf{r}}{dt^2}\right|_F = \left.\frac{d^2\mathbf{r}}{dt^2}\right|_M + 2\boldsymbol{\omega}\times\left.\frac{d\mathbf{r}}{dt}\right|_M + \boldsymbol{\omega}\times(\boldsymbol{\omega}\times\mathbf{r}_M) + \frac{d\boldsymbol{\omega}}{dt}\times\mathbf{r}_M$$

Consider a simple pendulum formed by a mass attached to the end of a string of length L, mounted to the ceiling of the physics building. In spherical coordinates defined with respect to the axis of rotation of the Earth, the physics building is located at the coordinates θ (the polar angle) and ϕ (the azimuthal angle). Assume that the period of the earth's rotation about its axis is much longer than the period of the pendulum when in an inertial frame.

C. [**13** points] Show that for small displacements from vertical defined by coordinates x and y in the horizontal plane of the physics building, the position of the mass satisfies

$$\ddot{x} - 2\omega\dot{y}\cos\alpha + \frac{g}{L}x = 0 \ , \quad \ddot{y} + 2\omega\dot{x}\cos\alpha + \frac{g}{L}y = 0$$

where ω is the angular frequency of the earths rotation.

D. [**13** points] Solve these coupled differential equations subject to the boundary conditions at $t = 0$

$$x = 0 \ , \quad \dot{x} = 0 \ , \quad y = A \ , \quad \dot{y} = 0$$