Spring 2003 Qualifying Examination - Electromagnetism

IMPORTANT: Answer ALL THREE Questions

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1. [**35** points total] (**Time Dependent Fields**)

The scalar and vector potentials resulting from an arbitrary localized source are

$$V(\mathbf{r},t) = \frac{1}{4\pi\varepsilon_0}\int d^3\mathbf{r}'\ \frac{\rho(\mathbf{r}',t_r)}{|\mathbf{r}-\mathbf{r}'|} \quad , \quad \mathbf{A}(\mathbf{r},t) \ = \ \frac{\mu_0}{4\pi}\int d^3\mathbf{r}'\ \frac{\mathbf{j}(\mathbf{r}',t_r)}{|\mathbf{r}-\mathbf{r}'|} \quad .$$

A. [**5** points] Define ρ, $\mathbf{j}$, $\mathbf{r}$, $\mathbf{r}'$. Define t_r in terms of $\mathbf{r}$, $\mathbf{r}'$, t, and the speed of light c.

After time $t = 0$ a time-dependent, uniform current density $\mathbf{K}(t)$ flows in the infinite xy-plane at $z = 0$, i.e. $\mathbf{K}(t) = \hat{\mathbf{e}}_y\ K_0(t)\delta(z)$, where $\delta(z)$ is a Dirac-delta function, $\hat{\mathbf{e}}_y$ is the unit-vector in the y-direction, and with $K_0(t) = 0$ for $t < 0$. There are no free charges on the sheet, or anywhere in space, at any time.

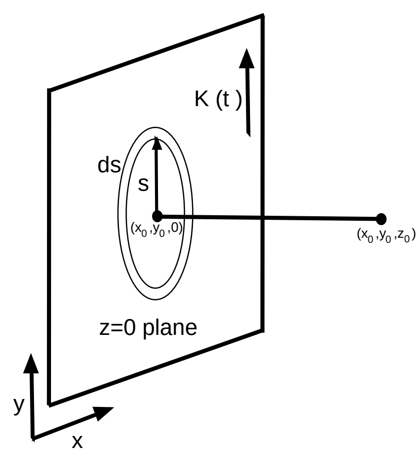

FIG. 1. *A current density* $\mathbf{K}(t)$ *flows in an infinite sheet located at* $z = 0$.

B. [**9** points] For a point located at $x = x_0$, $y = y_0$ and $z = z_0$ give the contributions to $V(x_0, y_0, z_0, t)$ and $\mathbf{A}(x_0, y_0, z_0, t)$ from the current in a ring of radius s and width ds in the xy-plane centered on $x = x_0$, $y = y_0$, $z = 0$.

C. [**3** points] At any given time, t, what is the maximum value of s that can contribute to $V(x_0, y_0, z_0, t)$ and $\mathbf{A}(x_0, y_0, z_0, t)$?

D. [**12** points] Show that

$$\mathbf{A}(x_0, y_0, z_0, t) = \hat{\mathbf{e}}_y\ \theta\left(t - \frac{z_0}{c}\right)\ \int_0^{t-\frac{z_0}{c}}\ d\eta\ f(\eta) \quad ,$$

where $\theta(x)$ is the step-function. Find $f(\eta)$.

E. [**6** points] Find the electric and magnetic fields for $z > 0$.

2. [**40** points total] (**Electrostatics**)

The most general solution to Laplace's equation, $\nabla^2 V = 0$, assuming azimuthal symmetry is

$$V(r,\theta) = \sum_{l=0}^{\infty} \left(A_l \, r^l + \frac{B_l}{r^{l+1}} \right) P_l(\cos\theta) \quad .$$

A conducting sphere of radius a at a potential V_0 is surrounded by a concentric non-conducting spherical shell of radius b with a surface charge density $\sigma(\theta) = K\cos\theta$, where K is constant.

- **A.** [**9** points] Write the allowed form of the potential in each of the two regions $a < r < b$ and $r > b$ without regard to the boundary conditions at $r = a$ and $r = b$.
- **B.** [**9** points] What is the discontinuity in the electric field across the non-conducting spherical shell? Explain the physical basis of your answer.
- **C.** [**8** points] What are the boundary conditions that must be satisfied by the potential $V(r,\theta)$?
- **D.** [**14** points] Find the potential in both regions.

3. [**25** points total] (**Energy and Momentum**)

Maxwell's equations and the Lorentz force law are

$$\nabla \cdot \mathbf{D} = \rho \ , \quad \nabla \times \mathbf{E} = -\frac{\partial}{\partial t}\mathbf{B} \ , \quad \nabla \cdot \mathbf{B} = 0 \ , \quad \nabla \times \mathbf{H} = \mathbf{J} + \frac{\partial}{\partial t}\mathbf{D}$$

$$\mathbf{F} = q \ (\mathbf{E} + \mathbf{v} \times \mathbf{B})$$

A. [**12** points] Starting from the Lorentz force law show that the work per unit time done on all charged particles in a system is given by

$$\frac{dW}{dt} = \int d^3\mathbf{r} \ \mathbf{E} \cdot \mathbf{J} \quad ,$$

where $\mathbf{J}$ denotes the current density of the charged particles.

B. [**8** points] A constant current I flows through a wire of finite resistance R that has a potential difference V maintained between its two ends. The wire has length L and radius a. Compute the electric and magnetic fields at the surface of the wire, and compute the Poynting vector. Comment on its direction.

C. [**5** points] Determine the electromagnetic energy per unit time entering the wire and compare this with naive expectations based on ohmic heating.