Spring 2002 Qualifying Exam – Quantum Mechanics

IMPORTANT: Answer ALL THREE Questions

1. [20 points total]

A. [5 points] A spinless particle moves freely in one dimension (1D) subject to a rectangular potential energy well

$$V(x) = \begin{cases} -V_0 & \text{for } x_0 \le x \le x_0 + L \\ 0 & \text{for } x < x_0 \text{ and } x > x_0 + L \end{cases}$$

Make a rough sketch of the wave function amplitude $|\Psi|$ as function of x for both the ground state and the first excited state, assuming both are bound states.

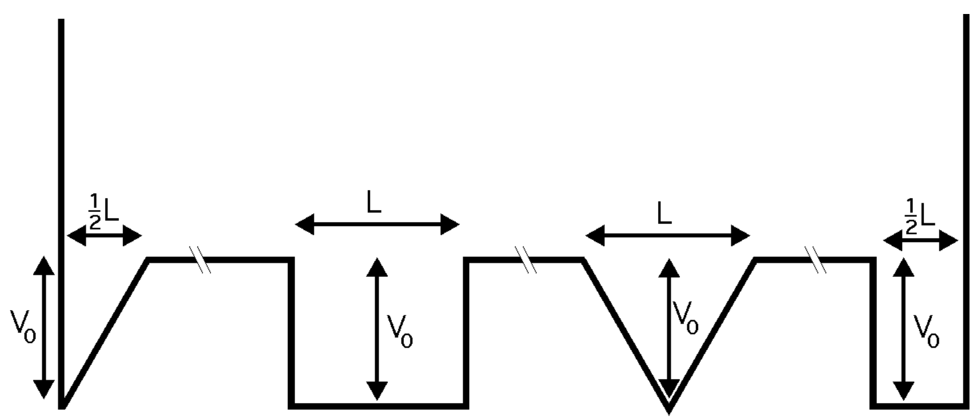
B. [5 points] Consider a parameter l_e to characterize the depth of the potential well as

$$V_0 = \frac{h^2}{2m}\frac{1}{l_e^2}$$

(m is the mass of the particle). Show that for $l_e = L/n$ there are at least n bound states.

C. [5 points] The figure below shows four potential wells. All are very far apart from each other. Two are at the edge, beyond which $V(x) = \infty$. Which well has the lowest bound state energy? Explain your answer with a qualitative argument.

D. [5 points] Derive an exact relation between the bound state energy levels of the triangular well in the middle and the semi-triangular well at the edge.



2. [50 points total]

Two spin-$\frac{1}{2}$ particles interact via the Hamiltonian

$$\hat{H} = \vec{\sigma}_1 \cdot \vec{\sigma}_2$$

where the unit of energy is chosen such that $\vec{\sigma}_1$ and $\vec{\sigma}_2$ are Pauli spin matrices.

A. [5 points] Rewrite the Hamiltonian in terms of

$$\hat{\sigma}^z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad , \quad \hat{\sigma}^- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \quad , \text{ and } \quad \hat{\sigma}^+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

B. [5 points] Rewite the Hamiltonian as a 4×4-matrix using the eigenstates of $\hat{\sigma}_1^z$ and $\hat{\sigma}_2^z$ as a basis.

C. [5 points] Find the energy levels and eigenstates of $\hat{H}$ using this 4×4-matrix.

D. [5 points] Write the rotation operator $\hat{R}(\alpha)$, which rotates the spins through an angle α about the x-axes, in terms of the Pauli-spin operators $\hat{\sigma}^x$, $\hat{\sigma}^y$, and $\hat{\sigma}^z$.

E. [5 points] Show that for $\alpha = \frac{1}{2}\pi$ your rotation operator reduces to

$$\hat{R}(\frac{1}{2}\pi) = \frac{1}{2}(1 - i\hat{\sigma}_1^x)(1 - i\hat{\sigma}_2^x)$$

F. [5 points] Show by explict evaluation that $[\hat{R}(\frac{1}{2}\pi), \hat{H}] = 0$.

G. [5 points] Rewrite $\hat{H}$ in terms of total angular momentum $\vec{J} = \vec{\sigma}_1 + \vec{\sigma}_2$.

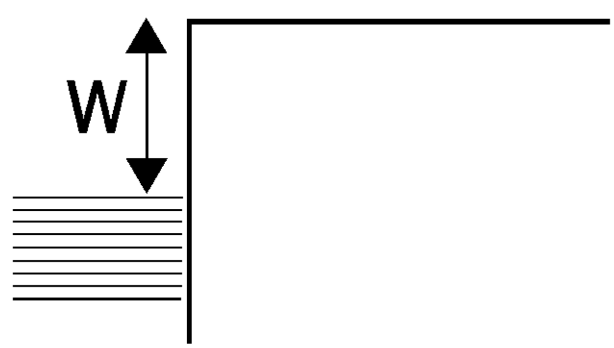
H. [5 points] Relate and identify the eigenstates of $\hat{H}$ in terms of those of $\hat{J}$.

I. [5 points] How do the eigenstates of $\hat{H}$ transform under rotations $\hat{R}(\frac{1}{2}\pi)$?

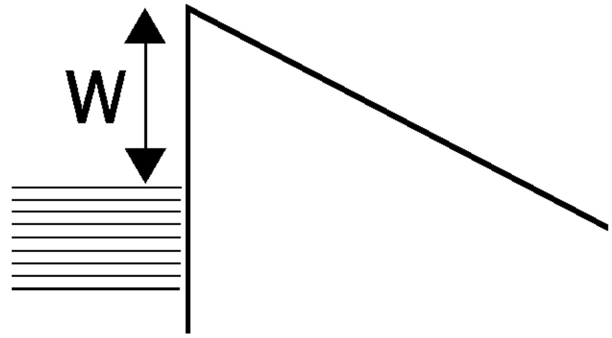
J. [5 points] What is the ground state expectation value of $\langle \sigma_1^x \sigma_2^x \rangle$?

3. [30 points total]

The energy levels near the surface of a metal appear roughly as shown in the figure to the right. The energy gap between the Fermi surface of the metal and the vacuum outside is the work function W. Incident photons with energies greater than W can free electrons from the metal surface.



The electron emission mechanism changes in the presence of an electric field $\mathcal{E}$ as shown in the figure to the right. Electrons can tunnel through the potential barrier. This process is known as "field emission".



A. [20 points] The major factor controlling the field emission rate is the WKB barrier penetration factor $|T|^2$. Show that

$$|T|^2 = \exp(-\frac{4}{3}\frac{W}{\hbar}\frac{\sqrt{2mW}}{e\mathcal{E}})$$

where m is the electron mass.

B. [10 points] Consider a metal for which the photo-electric threshold is 2 eV, corresponding to a light frequency $\nu_0 = 5 \times 10^{14}$ Hz. Estimate the magnitude of the electric field, $\mathcal{E}$ in volts per cm, needed to create a substantial field emission rate. The electron rest energy is equal to $mc^2 = 5 \times 10^5$ eV and the light speed is $c = 3 \times 10^{10}$ cm/s.