The eigenfunction's functions of the hydrogen atom obtained from nonrelativistic quantum mechanics have the form

$$\psi_{n\ell m}(\vec{x}) = R_{n\ell}(r) Y_\ell^m(\theta,\phi), \qquad n = \ell+1,\ \ell+2\,,\cdots$$

where $Y_\ell^m(\theta,\phi)$ are the spherical harmonics. The corresponding energy levels are given by the Bohr formula

$$E_n = -\left(\frac{e^2}{4\pi\epsilon_0}\right)^2 \frac{m}{2\hbar^2 n^2}\,, \tag{1}$$

where m and e are the mass and charge of the electron. (The proton is assumed to be infinitely heavy.) The $\ell = 0$, and $\ell = 1$ spherical harmonics are

$$Y_0^0(\theta,\phi) = \frac{1}{\sqrt{4\pi}}\,,$$

$$Y_1^{+1}(\theta,\phi) = -\sqrt{\frac{3}{8\pi}}\sin\theta e^{i\phi}\,, \qquad Y_1^{-1}(\theta,\phi) = \sqrt{\frac{3}{8\pi}}\sin\theta e^{-i\phi}\,, \qquad Y_1^0(\theta,\phi) = \sqrt{\frac{3}{4\pi}}\cos\theta\,.$$

1. [25 points]

   a) [5 points] Show that the states with $\ell = 1$, and $\ell = 0$ have a definite parity and determine the parity in each case.

   b) [20 points] Derive from the Schrodinger equation the above expression eq. (1) for the ground state energy E_1, and find the radial part of the ground state wave function $R_{10}(r)$ (unnormalized).

2. [50 points] A hydrogen atom is in a uniform electric field E in the $+z$ direction. In this problem you should express your answer in terms of integrals over the radial functions $R_{n\ell}(r)$.

   a) [25 points] Find the lowest order non-zero correction to the energy for a hydrogen atom in the ground state $n = 1$. Comment on the sign of the result and on the nature of the dependence of the energy on the electric field E.

   b) [25 points) Work out the lowest order corrections to the energies of the $n = 2$ states. What are the corresponding wave functions?
   Discuss how and why this result differs from that of part (a).

3. [25 points] Two identical non-interacting particles move in a one-dimensional potential.

a) [5 points] Write general symmetric and anti-symmetric two-particle wave functions in terms of two orthogonal one-particle wave functions.

b) [10 points] Show that the average of the square of the particle separation is larger if the particles have an anti-symmetric wave function than if they have a symmetric wave function.

c) [10 points] Consider the situation where the particles are electrons, and the potential is that due to two charges (protons) located at $x = \pm R/2$, where R is much greater than the "radius" of the electron in the ground state of the hydrogen atom. Use approximate one-electron ground state eigenfunctions localized near each of the two charges to construct symmetric and antisymmetric spatial two-electron wave functions for the hydrogen molecule. Based on the results of part (b), what is the total electron spin in the ground state of the hydrogen molecule?