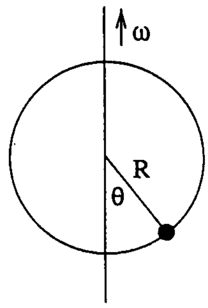
Classical Mechanics

$$\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$$
$$\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$$

1. (*45 points*) A circular wire hoop is rotating about a vertical axis (along a diameter) with constant angular velocity ω. A bead of mass m, is free to slide without friction on the hoop. A convenient definition is $\omega_0^2 = g/R$



a) 5 pts Draw all the forces (in the Lab frame) on the bead when it is in an equilibrium position (for $0 < \theta < \pi$). Also show the net force (label it clearly).

b) 5 pts Write the Lagrangian.

c) 10 pts Derive the equation of motion from the Lagrangian.

d) 10 pts Find the stable equilibrium position θ_q, of the bead as a function of ω.
There is a critical value of ω, ω_c, below which the nature of the equilibrium changes.
Find ω_c. Describe the nature of the change.

e) 15 pts Find the frequency of small oscillations of the mass about the equilibrium point θ_q.
Assume that the angular velocity is above the critical value ω_c and $0 < \theta_q < \pi$.

2. (*25 points*) The inertia tensor of an object composed of point masses m_p, is

$$I_{ij} = \sum_p m_p \left(\delta_{ij} \sum_{k=1}^{3} (x_p^k)^2 - x_p^i x_p^j \right)$$

a) 10 pts Show that

$$I_n = \hat{n} \cdot \mathbf{I} \cdot \hat{n}$$

is the moment of inertia about an axis specified by the unit vector $\hat{n}$.
$\hat{n}$ is a unit vector pointing in an arbitrary direction.

b) 10 pts If $I_{ij} = I_i$ is diagonal and $\vec{\omega}$ is in an arbitrary direction derive an explicit expression for the angular momentum $\vec{L}$ in terms of the components of ω and I_i.
Work this out using subscript notation.

c) 5 pts Under what general conditions will the angular momentum vector point in the direction of the angular velocity vector?

3. (*30 points*) You wish to send a spacecraft from Earth to Jupiter via a fuel-efficient *Hohmann* transfer orbit. This is an elliptical orbit around the sun tangent to both the Earth's orbit and to Jupiter's orbit.

Assume that the orbits of Earth and Jupiter are circular and coplanar and ignore other planets as well as the gravitational attraction of the spacecraft to Earth and Jupiter.

Call the radius of Jupiter's orbit, R_J and the radius of Earth's orbit, $R_e = 1$ AU.

For general information (you don't need this number) $R_J \approx 5.2$ AU. The velocity of the earth in its orbit is $v_e = 2\pi$ AU/year.

a) $^{10\ pts}$ A relationship between the semi-major axis a, of the ellipse and the total energy E, is

$$a = \frac{\gamma m}{2|E|}$$

where $\gamma = GM_s$ and m is the mass of the orbiting object.

Derive this for the special case of a circular orbit.

b) $^{15\ pts}$ What speed does the spacecraft have when it reaches the orbit of Jupiter? Give your answer in terms of the speed of Jupiter in its orbit v_J, and the orbit radii, R_J and R_e.

Is the spacecraft speed larger or smaller than the speed of Jupiter in its orbit?

c) $^{5\ pts}$ How does the speed of the spacecraft when it reaches Jupiter's orbit compare with its speed when it started its elliptical path at Earth's orbit (give the answer in terms of orbit radii)?