Selected Mathematical Formulae

Differential Operators

$$\begin{aligned}
\boldsymbol{\nabla}\psi &= \frac{\partial\psi}{\partial x}\,\hat{\boldsymbol{e}}_x + \frac{\partial\psi}{\partial y}\,\hat{\boldsymbol{e}}_y + \frac{\partial\psi}{\partial z}\,\hat{\boldsymbol{e}}_z && \text{(cartesian)}\\
&= \frac{\partial\psi}{\partial\rho}\,\hat{\boldsymbol{e}}_\rho + \frac{1}{\rho}\frac{\partial\psi}{\partial\phi}\,\hat{\boldsymbol{e}}_\phi + \frac{\partial\psi}{\partial z}\,\hat{\boldsymbol{e}}_z && \text{(cylindrical)}\\
&= \frac{\partial\psi}{\partial r}\,\hat{\boldsymbol{e}}_r + \frac{1}{r}\frac{\partial\psi}{\partial\theta}\,\hat{\boldsymbol{e}}_\theta + \frac{1}{r\sin\theta}\frac{\partial\psi}{\partial\phi}\,\hat{\boldsymbol{e}}_\phi && \text{(spherical)}
\end{aligned}$$

$$\begin{aligned}
\nabla^2\psi &= \frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2} + \frac{\partial^2\psi}{\partial z^2}\\
&= \frac{1}{\rho}\frac{\partial}{\partial\rho}\Big(\rho\,\frac{\partial\psi}{\partial\rho}\Big) + \frac{1}{\rho^2}\frac{\partial^2\psi}{\partial\phi^2} + \frac{\partial^2\psi}{\partial z^2}\\
&= \frac{1}{r^2}\frac{\partial}{\partial r}\Big(r^2\frac{\partial\psi}{\partial r}\Big) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\Big(\sin\theta\,\frac{\partial\psi}{\partial\theta}\Big) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2\psi}{\partial\phi^2}
\end{aligned}$$

$$\begin{aligned}
\boldsymbol{\nabla}\cdot\boldsymbol{A} &= \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}\\
&= \frac{1}{\rho}\frac{\partial}{\partial\rho}\,(\rho A_\rho) + \frac{1}{\rho}\frac{\partial A_\phi}{\partial\phi} + \frac{\partial A_z}{\partial z}\\
&= \frac{1}{r^2}\frac{\partial}{\partial r}\,(r^2 A_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\,(\sin\theta A_\theta) + \frac{1}{r\sin\theta}\frac{\partial A_\phi}{\partial\phi}
\end{aligned}$$

$$\begin{aligned}
\boldsymbol{\nabla}\times\boldsymbol{A} &= \Big(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\Big)\,\hat{\boldsymbol{e}}_x + \Big(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\Big)\,\hat{\boldsymbol{e}}_y + \Big(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\Big)\,\hat{\boldsymbol{e}}_z\\
&= \Big[\frac{1}{\rho}\frac{\partial A_z}{\partial\phi} - \frac{\partial A_\phi}{\partial z}\Big]\,\hat{\boldsymbol{e}}_\rho + \Big[\frac{\partial A_\rho}{\partial z} - \frac{\partial A_z}{\partial\rho}\Big]\,\hat{\boldsymbol{e}}_\phi + \frac{1}{\rho}\,\Big[\frac{\partial}{\partial\rho}(\rho A_\phi) - \frac{\partial A_\rho}{\partial\phi}\Big]\,\hat{\boldsymbol{e}}_z\\
&= \frac{1}{r\sin\theta}\Big[\frac{\partial}{\partial\theta}(\sin\theta A_\phi) - \frac{\partial A_\theta}{\partial\phi}\Big]\,\hat{\boldsymbol{e}}_r + \Big[\frac{1}{r\sin\theta}\frac{\partial A_r}{\partial\phi} - \frac{1}{r}\frac{\partial}{\partial r}(rA_\phi)\Big]\,\hat{\boldsymbol{e}}_\theta + \frac{1}{r}\Big[\frac{\partial}{\partial r}(rA_\theta) - \frac{\partial A_r}{\partial\theta}\Big]\,\hat{\boldsymbol{e}}_\phi
\end{aligned}$$

Spherical Harmonics

$$\int d\Omega\, Y^*_{\ell m}(\hat{\boldsymbol{n}})\,Y_{\ell' m'}(\hat{\boldsymbol{n}}) = \delta_{\ell\ell'}\,\delta_{mm'}\,, \qquad \frac{4\pi}{2\ell+1}\sum_{m=-\ell}^{\ell} Y^*_{\ell m}(\hat{\boldsymbol{n}}')\,Y_{\ell m}(\hat{\boldsymbol{n}}) = P_\ell(\hat{\boldsymbol{n}}'\cdot\hat{\boldsymbol{n}})$$

$$Y_\ell^m(\theta,\phi) = \sqrt{\frac{2\ell+1}{4\pi}\frac{(\ell-m)!}{(\ell+m)!}}\;P_\ell^m(\cos\theta)\,e^{im\phi}$$

$$Y_0^0(\theta,\phi) = \sqrt{\frac{1}{4\pi}}\,,\quad Y_1^0(\theta,\phi) = \sqrt{\frac{3}{4\pi}}\,\cos\theta\,,\quad Y_1^{\pm1}(\theta,\phi) = \mp\sqrt{\frac{3}{8\pi}}\,\sin\theta\,e^{\pm i\phi}$$

Pauli matrices

$$\sigma_x = \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}, \qquad \sigma_y = \begin{pmatrix}0 & -i\\ i & 0\end{pmatrix}, \qquad \sigma_z = \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}$$

Clebsch-Gordon Coefficients

$$|JM\rangle = \sum_{m_1=-j_1}^{j_1} \sum_{m_2=-j_2}^{j_2} |j_1\, m_1\rangle \otimes |j_2\, m_2\rangle\; \langle j_1\, m_2; j_2\, m_2|JM\rangle$$

$$\langle j_1 \, {-m_2};\; j_2 \, {-m_2}|J \, {-M}\rangle = (-1)^{j_1+j_2-J}\langle j_1\, m_2; j_2\, m_2|JM\rangle$$

$$\langle \tfrac{1}{2} \, {+\tfrac{1}{2}};\; \tfrac{1}{2} \, {+\tfrac{1}{2}}|1\, 1\rangle = 1\,, \qquad \langle \tfrac{1}{2} \, {+\tfrac{1}{2}};\; \tfrac{1}{2} \, {-\tfrac{1}{2}}|1\, 0\rangle = \tfrac{1}{\sqrt{2}}\,, \qquad \langle \tfrac{1}{2} \, {+\tfrac{1}{2}};\; \tfrac{1}{2} \, {-\tfrac{1}{2}}|0\, 0\rangle = \tfrac{1}{\sqrt{2}}$$

$$\langle \tfrac{1}{2} \, {-\tfrac{1}{2}};\; \tfrac{1}{2} \, {-\tfrac{1}{2}}|1 \, {-1}\rangle = 1\,, \qquad \langle \tfrac{1}{2} \, {-\tfrac{1}{2}};\; \tfrac{1}{2} \, {+\tfrac{1}{2}}|1\, 0\rangle = \tfrac{1}{\sqrt{2}}\,, \qquad \langle \tfrac{1}{2} \, {-\tfrac{1}{2}};\; \tfrac{1}{2} \, {+\tfrac{1}{2}}|0\, 0\rangle = -\tfrac{1}{\sqrt{2}}$$

Integral Theorems

$$\int_V d^3x\; \boldsymbol{\nabla}\psi = \int_S da\, \boldsymbol{n}\, \psi \qquad \int_S da\, \boldsymbol{n} \times \boldsymbol{\nabla}\psi = \oint_C d\boldsymbol{l}\, \psi$$

$$\int_V d^3x\; \boldsymbol{\nabla} \cdot \boldsymbol{A} = \int_S da\, \boldsymbol{n} \cdot \boldsymbol{A} \qquad \int_S da\, \boldsymbol{n} \cdot (\boldsymbol{\nabla} \times \boldsymbol{A}) = \oint_C d\boldsymbol{l} \cdot \boldsymbol{A}$$

$$\int_V d^3x\; \boldsymbol{\nabla} \times \boldsymbol{A} = \int_S da\, \boldsymbol{n} \times \boldsymbol{A}$$

$$\int_V d^3x\; (\phi\nabla^2\psi + \boldsymbol{\nabla}\phi \cdot \boldsymbol{\nabla}\psi) = \int_S da\, \boldsymbol{n} \cdot (\phi\boldsymbol{\nabla}\psi)$$

$$\int_V d^3x\; (\phi\nabla^2\psi - \psi\nabla^2\phi) = \int_S da\, \boldsymbol{n} \cdot (\phi\boldsymbol{\nabla}\psi - \psi\boldsymbol{\nabla}\phi)$$

where $da\,\boldsymbol{n} \equiv d\boldsymbol{a}$ is the differential area element with direction normal to the surface S.

Vector Identities

$$\boldsymbol{A} \cdot (\boldsymbol{B} \times \boldsymbol{C}) = \boldsymbol{B} \cdot (\boldsymbol{C} \times \boldsymbol{A}) = \boldsymbol{C} \cdot (\boldsymbol{A} \times \boldsymbol{B})$$

$$\boldsymbol{A} \times (\boldsymbol{B} \times \boldsymbol{C}) = (\boldsymbol{A} \cdot \boldsymbol{C})\, \boldsymbol{B} - (\boldsymbol{A} \cdot \boldsymbol{B})\, \boldsymbol{C}$$

$$(\boldsymbol{A} \times \boldsymbol{B}) \cdot (\boldsymbol{C} \times \boldsymbol{D}) = (\boldsymbol{A} \cdot \boldsymbol{C})(\boldsymbol{B} \cdot \boldsymbol{D}) - (\boldsymbol{A} \cdot \boldsymbol{D})(\boldsymbol{B} \cdot \boldsymbol{C})$$

$$\boldsymbol{\nabla} \cdot (\psi\boldsymbol{A}) = \boldsymbol{A} \cdot \boldsymbol{\nabla}\psi + \psi\, \boldsymbol{\nabla} \cdot \boldsymbol{A}$$

$$\boldsymbol{\nabla} \times (\psi\boldsymbol{A}) = (\boldsymbol{\nabla}\psi) \times \boldsymbol{A} + \psi\, \boldsymbol{\nabla} \times \boldsymbol{A}$$

$$\boldsymbol{\nabla}(\boldsymbol{A} \cdot \boldsymbol{B}) = (\boldsymbol{A} \cdot \boldsymbol{\nabla})\boldsymbol{B} + (\boldsymbol{B} \cdot \boldsymbol{\nabla})\boldsymbol{A} + \boldsymbol{A} \times (\boldsymbol{\nabla} \times \boldsymbol{B}) + \boldsymbol{B} \times (\boldsymbol{\nabla} \times \boldsymbol{A})$$

$$\boldsymbol{\nabla} \cdot (\boldsymbol{A} \times \boldsymbol{B}) = \boldsymbol{B} \cdot (\boldsymbol{\nabla} \times \boldsymbol{A}) - \boldsymbol{A} \cdot (\boldsymbol{\nabla} \times \boldsymbol{B})$$

$$\boldsymbol{\nabla} \times (\boldsymbol{A} \times \boldsymbol{B}) = \boldsymbol{A}(\boldsymbol{\nabla} \cdot \boldsymbol{B}) - \boldsymbol{B}(\boldsymbol{\nabla} \cdot \boldsymbol{A}) + (\boldsymbol{B} \cdot \boldsymbol{\nabla})\boldsymbol{A} - (\boldsymbol{A} \cdot \boldsymbol{\nabla})\boldsymbol{B}$$

$$\boldsymbol{\nabla} \times \boldsymbol{\nabla}\psi = 0$$

$$\boldsymbol{\nabla} \cdot (\boldsymbol{\nabla} \times \boldsymbol{A}) = 0$$

$$\boldsymbol{\nabla} \times (\boldsymbol{\nabla} \times \boldsymbol{A}) = \boldsymbol{\nabla}(\boldsymbol{\nabla} \cdot \boldsymbol{A}) - \nabla^2\boldsymbol{A}$$